Adaptive finite elements for a contact problem in elastoplasticity with Lagrange techniques

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Abstract

The topic of this thesis is the derivation and analysis of some finite element schemes for a contact problem in elastoplasticity. These schemes are based on the formulation of the models as saddle point problems and use finite element spaces of arbitrary polynomial degrees. In this thesis, these new approaches with higher-order finite elements are shown to be well defined and convergent. Moreover, some a priori estimates on the rates of convergences are proven. The use of Lagrange multipliers in the saddle point formulation yields a coherent approach to reliable a posteriori error estimates for the proposed higher-order schemes. Additionally, the Lagrange multipliers are used to show the equivalence of the errors of the stresses and the energies, for low order finite elements using triangular or quadrilateral cells. For the first time, this allows for a proof of convergence for quadrilateral-based adaptive finite elements. Furthermore, the approach based on triangular cells is shown to be of optimal convergence. The theoretical findings are confirmed by numerical experiments.

Zusammenfassung

Das Thema dieser Dissertation ist die Herleitung und numerische Analyse von finiten Elementen für ein Problem in der Elastoplastizität mit Kontaktbedingungen. Die hergeleiteten finite Elemente Verfahren basieren auf einer Formulierung als Sattelpunktproblem und der Nutzung von Polynomen höherer Ordnung. Die Analyse der vorgestellten Verfahren beginnt mit dem Zeigen der Wohldefiniertheit und der Konvergenz. Im nächsten Schritt werden a priori Abschätzungen der Konvergenzraten gezeigt. Weiterhin führt die Einführung von Lagrange Multiplikatoren zu einem einheitlichen Ansatz zur a posteriori Abschätzung des Diskretisierungfehlers unter der Verwendung von Elementen höherer Ordnung. Zusätzlich ermöglicht es der Zugang über Lagrange Multiplikatoren die Äquivalenz der Diskretisierungsfehler in den Spannungen und in den Energien für finite Elemente niederer Ordnung zu zeigen, was insbesondere neu für Viereckselemente ist. Diese Äquivalenz wiederum erlaubt nun den Beweis der Konvergenz von adaptiven finiten Elementen niederer Ordnung. Für Dreieckselemente wird sogar die optimale Konvergenz bewiesen. Die theoretischen Erkenntnisse werden durch numerische Experimente bestätigt.

Contents

Introduction

The topic of this thesis is the derivation and analysis of finite element methods for a contact problem with friction in small strain elastoplsticity. Within this analysis existence and convergence results are shown. In particular, a posteriori error estimates and an adaptive finite element method are introduced. The introduction to this thesis is divided into a short overview of the background followed by an overview of the content.

Background

For many applications in mechanical and civil engineering the prediction of the behavior of complicated mechanical systems plays an important role. When interpreting and processing these predictions, several inevitable errors have to be kept in mind. First of all, accurate models of the systems have to be derived. To do this, the single components of a system are usually idealized. Hence, there remains a so-called modeling error between measurements and the predictions of the model. Furthermore, differences between reality and model arise from measurement errors. However, the modeling and measurement errors are not the subject of this work.

Instead, the focus is on another important source of error between the behavior of a system and its computer simulation. It usually rises since even the ideal model's exact analytic solution is not known in general. Hence, it is necessary to determine appropriate approximations of the analytic solutions. The reduction of the the resulting error is the motivation of this thesis. The idealized models are usually formulated in an infinite dimensional framework. A discretization is thus necessary in order to actually calculate an approximate solution on computers. The decision whether the approximation is appropriate has to be based on a rigorous mathematical analysis.

A typical approach to obtain a discretization consists in the replacement of the infinite dimensional spaces by finite dimensional ones. The result is a problem in a finite dimensional framework which can be solved on computers. The solution of the finite dimensional problems is called the discrete solution. The distance between the analytic and the exact discrete solution is referred to as the discretization error. Errors introduced in numerical schemes for example due to rounding and iterative procedures are neglected in this thesis. For the scheme to be appropriate, the discrete solution should at least converge to the analytic one as the dimension of the discrete problem approaches infinity. The speed of convergence of the discretization error is often used to judge the quality of the approximation scheme.

Finite element spaces are among the most common discretization spaces [22, 21]. Their definition is based on the segmentation of the considered domain into smaller domains of easy-to-handle geometry. In two space dimensions, these small domains or cells are usually triangles or quadrilaterals. Hexahedrons, tetrahedrons, and pyramids are among the most common three dimensional cells. The basis of the discrete spaces often consist of piecewise polynomial functions. The increase of the dimension of the discrete space can be achieved by the use of more and smaller cells or of polynomials with higher degrees. The former approach is known as h-refinement and the latter as p-refinement. Moreover, uniform refinement usually consists in the division of all cells into smaller cells or the increase of the piecewise polynomial degree for all cells of the mesh in the same way. If the refinement is uniform, the speed of convergence of finite element methods is usually stated in terms of the diameter of cells and the polynomial degrees of the basis functions.

The possible convergence rate usually depends on the regularity of the analytic solution. Less regularity results in slower convergence. A well-established way to recover faster convergence are so-called adaptive finite element methods (AFEM). In AFEM, only parts of the mesh are refined. The decision whether a cell is refined or not is usually based on so-called a posteriori error estimators. Such estimators are upper bounds of the discretization error and are computed from the discrete solution. We refer to the monographs [107, 2, 8] for an overview on a posteriori error control for linear problems. However, the elastoplastic and contact problems dealt with in this work are nonlinear and the development of adaptive schemes has to be adjusted. The development and analysis of a posteriori estimates is one focal point of this thesis.

Models of elastoplastic material behavior predict the response of a deformable body under load. A material is called elastoplastic if after a phase of elastic response some of the deformation still remains after the exterior forces no longer act on the body. A hardening law is present if the stress within the material further increases in the plastic deformation phase [68, 57]. In the absence of such a hardening law, the stress does not increase further. This behavior is referred to as perfect plasticity [104]. The task to find the displacement, the plastic strain, and other internal variables is called the primal problem [57]. In the dual problem, the stress is an additional unknown variable itself. The weak form of the primal problem results in a variational inequality which is sometimes called variational inequality of second kind [19]. The mathematical analysis of the properties of the primal problem with hardening is usually based on results of convex analysis, as found for example in [49]. The studies of the problem have reached a certain maturity [101, 57]. Though, there are still more recent results on the regularity of its solution, e.g. [79, 71]. The fundamentals of the numerical analysis of the primal problem can be found in the monograph [57]. A derivation of adaptive finite elements based on a posteriori error estimates is presented in [4, 19] for a residual type estimator, in [81] for a dual weighted residual approach, in [39] for an equilibrated estimate, in [29] an averaging estimator is considered and some results on a functional estimator are found in [85].

The efficient computation of an approximate solution for problems in elastoplasticity encounters several difficulties. See e.g. [52] for an overview of standard methods for nonlinear problems. For materials with hardening, the main problem lies in the treatment of a non differentiable dissipation functional [101]. In the Literature, many numerical solution schemes for problems in elastoplasticity are proposed ranging form classical radial return [57, 101] to Newton method like schemes [7, 44, 53]. For perfect plastic material laws, the derivation of solvable discrete approximation schemes becomes more important [109]. An example of an implementation for some type of elastoviscoplasticity is given in [35]. Furthermore, Carstensen [27] showed the convergence of specific adaptive finite element methods for an elastoplastic and certain other nonlinear problems. This thesis focuses on the derivation of error estimates and the convergence analysis for contact problems in elastoplsaticity.

In mechanical engineering problems, the load is often not completely given as a force but results from the contact with another object [23, 6, 46, 45, 15]. For linear elastic material laws, the contact conditions often restrict only the displacement of the considered body and the most common of them is the Signorini problem. The application of such contact conditions to elastoplastic materials is straight forward. The modeling of contact problems in linear elasticity leads to another type of variational inequality [69, 64], sometimes referred to as of first kind. The difference to the nonlinearity in elastoplasticity is the fact that the contact conditions restrict the set of admissible displacements rather than introduce a non differentiable functional.

Another important aspect of structural mechanics is the consideration of friction phenomena. The monograph [112] introduces frictional contact for elastoplastic materials from a mechanical engineering perspective. Whereas, a more mathematical view can be found in [54]. A simple but widely applicable model is Coulomb's friction law. In this setting the frictional resistance is assumed to be given by the product of a problem dependent constant and the normal force. The case where the friction resistance is given is referred to as Tresca friction. It can be used within a simple fixed point iteration to approximate Coulomb's friction law [48]. As for the contact conditions the friction conditions can be applied directly to the elastoplastic material law. Additionally, the numerical analysis for elastoplastic contact problems can be performed within the framework known from the respective problems.

The numerical approximation of the solution of contact problems with friction is of great importance in computational mechanics [112]. The monographs [69, 64] give a good introduction to the numerical aspects of contact problems from a mathematical view. For linear elastic materials, efficient numerical solution schemes based on multigrid methods are proposed for example in [73, 111]. A numerical solution scheme for elastoplastic contact is proposed, for example in [54]. For quasistatic contact problems in viscoplasticity more details can be found in [59, 50].

The considered discretizations use different numbers of Lagrange multipliers for contact, friction and the dissipation functional as well as relatively arbitrary spaces. The analysis of the discrete problems establishes results on the existence and uniqueness of discrete solutions and their convergence to the analytic one. Moreover, as one main result of this work, this analysis yields a priori as well as a posteriori estimates on the discretization error. In a first step, these bounds are each stated for mostly arbitrary discrete spaces. Next, the use of specific finite element spaces of polynomials yields more detailed estimates based on the previously established more general ones. Subsequently, the a posteriori estimator gives rise to an adaptive finite element scheme. The next main result of this thesis consists in the proof of convergence for two of the discretization approaches of the elastoplastic problem without contact conditions. One is based on meshes of triangles whereas the other uses quadrilaterals. For the approach based on triangles even quasi optimal convergence is shown.

Outline

In Chapter 1, the material models are presented which will be discretized in the following chapters. The derivation of the material laws is well known, cf. [57, 101], and therefore details are omitted. The chapter opens with a brief introduction of the classic linear elastic material law, for details see for example [78]. This model is extended to the primal problem of rate independent small strain elastoplasticity with linear hardening as found in e.g. [55, 101, 26]. There exist many more additional approaches to model various elastoplastic behavior, for example viscoplasticity and finite strain elastoplasticity see [101]. In the primal problem considered in this thesis, the sought unknowns are the displacement and the plastic strain as well as additional internal variables. The number of unknowns depends on the hardening law. In the case of perfect plasticity, the numerical analysis and solution of the problem differs from the one with hardening. However, perfect plastic materials will not be considered, for details see for example [66, 109]. This thesis focuses on the primal problem with hardening.

The weak formulation of the primal problem of elastoplasticity with hardening is given in terms of a variational inequality. The time discretization with an implicit Euler scheme results in a quasistatic problem which is equivalent to a nonlinear convex minimization problem. The existence and uniqueness of a minimizer follows from standard arguments of convex analysis as shown in [49]. A general approach of a mixed formulation based in Lagrange multipliers as given in [49] is adopted to the elastoplastic problem. This results in a saddle point problem with a similar structure as known from friction problems, see [64, 69]. Subsequently, an equivalent stationary condition for a saddle point is presented. It consists of a variational equality and a variational inequality.

In a next step, the elastoplastic material law is extended to include frictionless contact conditions. In the Signorini problem known from linear elasticity the contact conditions impose restrictions on the displacement. Hence, they are easily transfered to quasistatic elastoplasticity. The same way as for linear elastic contact problems mixed formulations of the elastoplastic contact problem are derived by the introduction of Lagrange multipliers. Two cases of mixed formulations are considered. One only treats the contact condition with Lagrange multiplier and uses a direct approach for the elastoplastic nonlinearity whereas in the other two Lagrange multipliers are included. However, if the nonlinear plastic dissipation functional is not treated by an additional Lagrange multiplier the stationary condition still consists of two variational inequalities. This formulation is also written as one variational equality and two variational inequalities. Nevertheless, both stationary conditions of the two considered formulations are equivalent in the sense that their primal solutions and their multipliers for the contact condition are the same, respectively. Moreover, each of the two formulations has exactly one saddle point since an inf-sup condition is shown to hold for the contact multiplier and the set of multipliers for plasticity is bounded.

In the last section of Chapter 1, an additional side condition which accounts for friction is imposed. In the two mixed formulations, the introduction of another Lagrange multiplier yields saddle point problems with two and three multipliers, respectively. The two formulations are equivalent to the same convex minimization problem and therefore the primal solution is the same for both. Additionally, it is shown that the Lagrange multipliers present in both formulations, i.e., the multipliers for contact and friction, are equal. The inf-sup condition as well as the boundedness of the sets of multipliers for friction and plasticity yield unique existence of saddle points for both problems.

The introduction and analysis of discretizations for the problems of the first chapter are the topic of Chapter 2. The main results of this section concern the existence and uniqueness as well as the convergence of discrete solutions. The models for elastoplasticity without side conditions or, alternatively, with frictional contact conditions are considered. Nevertheless, they implicitly included the model where only the contact condition and no friction is present. First, arbitrary finite dimensional subspaces are introduced to replace the infinite dimensional one in the weak formulation of the primal problem of elastoplasticity without contact. The result is a discrete variational inequality, and its solution is the unique minimizer of the energy functional over the discrete space. Subsequently, the saddle point problem is discretized in the same way, i.e., the set of multipliers is replaced by a subset of a discrete space. However, the set of the discrete Lagrange multipliers is allowed to be non conform in the sense that it does not have to be a subset of the set of analytic multipliers. Hence, the discrete saddle point problem and the discrete variational inequality are no longer equivalent. Nevertheless, the discrete saddle point problem is shown to have a unique solution.

Next, the discretization of the frictional contact problem leads to similar observations. Again, the nonconformity of the Lagrange multipliers yields the same results with respect to the equivalence of its discrete formulations. Though the uniqueness of the discrete Lagrange multiplier does not follow directly. A sufficient discrete inf-sup condition is introduced for the multipliers associated to contact and friction. The condition is the same as the one known from the discretization of the classic Signorini problem, cf. [62, 64].

The density of a sequence of subspaces turns out to be sufficient for the convergence of the discrete solution to the corresponding analytic solution. This holds for both the elastoplastic problem without further side conditions as well as the one with frictional contact conditions. For both cases, the first component of the saddle point is shown to converge strongly whereas the Lagrange multipliers are only shown to converge weakly. For the problem without contact conditions and without Lagrange multipliers, convergence results can also be found in [58]. And for the saddle point approach to contact problems in linear elasticity, similar convergence results are shown in [64].

The assumptions on the discrete spaces made for the proofs of existence and convergence are shown to be met by standard finite element spaces of arbitrary polynomial degrees. The meshes are allowed to contain quadrilaterals and/or triangles in two dimensions and hexahedrons in three dimensions. The approach is new for the mixed formulation of elastoplasticity without contact conditions. Usually, triangles and piecewise affine spaces are used in a direct approach [26, 35]. Additionally, the inclusion of contact conditions within a mixed formulation is a new result which is obtained in this chapter. For the Signorini problem, there already exist mixed formulations with polynomials of arbitrary degrees [92]. However, the inf-sup condition is shown to hold only under the assumption that a different mesh can be used for both multipliers defined on the contact boundary. Again, this is a new result for elastoplastic behavior which is already known for linear elastic materials [62, 61, 92].

Chapter 3 deals with the a priori quantification of the discretization error and its convergence speed. It starts with the derivation of error estimates based on arbitrary spaces. Together with the density assumptions, these estimates prove the strong convergence of the discrete Lagrange multipliers to the analytic ones. Again, the results are shown for the model of elastoplasticity with and without frictional contact conditions. Moreover, the mixed formulations from above with differing numbers of Lagrange multipliers are considered.

In a standard manner, interpolation results for piecewise polynomials are used to specify rates of convergence from the general a priori results. These rates depend on the used polynomial degrees and on the in general unknown regularity of the analytic solution. Results regarding the regularity of solutions in elastoplasticity can be found for example in [100, 79, 71].

The topic of Chapter 4 is the derivation of a posteriori estimates for the discretization error. A residual and a dual weighted residual approach are considered. First, a result for the discretization by affine functions of the problem without contact is presented which is already known from [4, 19] and was extended to a bilinear mixed discretization in [97]. Moreover, the estimate is known to still hold for an affine ansatz for two body contact problems without Lagrange multipliers, cf. [113]. In a next step of this chapter, the error estimator for the affine ansatz is shown to hold for some discretizations based on meshes of quadrilaterals.

Another new result is the extension of a posteriori estimate to the discrete saddle point problem for spaces of arbitrary polynomial degrees. The new estimator has to account for the nonconformity of the Lagrange multipliers. This novel approach follows the idea of [94] which itself is loosely based on the approach presented in [20]. The application of Lagrange multipliers for contact and plastic dissipation allows for an easy adaption of the techniques known for contact problems in linear elasticity, cf. [92]. Thus, the residual estimators for contact and elastoplasticity are combined to yield a new estimator for elastoplastic contact problems.

In the same way as for the residual estimator, the dual weighted residual estimates for the Signorini problem with friction found in [96] are transfered to the contact problem with elastoplastic material. The estimator includes the Lagrange multipliers and frictional contact whereas the dual weighted residual estimator as found in [81] does not.

The discussion of some adaptive finite element method and its convergence is the emphasis of Chapter 5. The convergence results are proven for two low order finite element discretizations of the primal problem of elastoplasticity without restrictions due to contact. Nevertheless, any of the introduced error estimators can be used within an adaptive finite element algorithm for the respective problem.

The first approach shown to be convergent uses piecewise affine and piecewise constant basis functions on triangles. The second one is based on piecewise bilinear and piecewise affine functions over quadrilaterals. Both approaches provide the pointwise exactness of the material law. This is a crucial point in the convergence analysis. The proof of convergence for the adaptive affine-constant ansatz is found in [40, 27, 41] whereas the convergence for the bilinear-affine ansatz is a newly obtained result.

Another main result of the chapter is the proof of optimal convergence for the affine-constant AFEM. The proof relies on the well established techniques for variational equalities, cf. [103, 42]. It uses the properties of a Dörfler type marking [47] and of the newest vertex bisection refinement rules [16].

In Chapter 6, some numerical examples are discussed. The examples illustrate the applicability of theoretical findings. Eventually, the thesis concludes with some remarks on remaining open questions and possible future work.

1 Models

In this chapter, we present variational formulations which model linear elasticity and elastoplastic materials, respectively. Moreover, we introduce contact problems for elastoplastic material behavior. Throughout this work, we focus on elastoplasticity with hardening.

In engineering problems, the description of contact and elastoplastic phenomena often has to be included in models for static or dynamic mechanical systems, see e.g. [112, 46, 6]. In Section 1.4 we present how the contact of a deformable body and a rigid object can be described. The contact conditions are included in the energy minimization problems of sections 1.2 and 1.3 by searching the minimum over a convex and closed subset $K \subset H^1_D(\Omega)$ rather than the whole space. The minimization can be reformulated to an equivalent saddle point problem in which the definition of the set of Lagrange multipliers involves the restrictions. In this so-called mixed formulation the constraints on the displacement are only satisfied in a weak sense, cf. [62, 60]. However, the discretization of the saddle point problem leads to a new approximation scheme in the sense that its discrete solution in general is not the solution of the discrete variational inequality.

For elastoplasticity with linear kinematic hardening this approach is also possible. The mixed formulation with a Lagrange multiplier gives rise to a discrete approach which is non conform for higher polynomial degrees. This non conformity results in a sequence of discrete approximations which do not solve the discrete analogon of the variational inequality. The higher order approximation scheme based on Lagrange multipliers is a new approach which will be presented in Chapter 2. Additionally, the mixed formulation naturally occurs when using Uzawa's method to solve the problem, see e.g. [55]. Though, Uzawa's method may not be the first choice for efficiently solving elastoplastic problems, for some details on solution schemes see e.g. [44]. Nevertheless, in the conform setting of lower polynomial degrees we can use the stationary condition to show the convergence of an adaptive finite element method based on quadrilaterals. For the scheme based on triangles the convergence result was already shown in [40]. However, the proof in Chapter 5 also holds for the discretization from [40] but does not use Jensen's inequality. Furthermore, the mixed formulation is also useful in the a posteriori analysis of the complete variational inequality of elastoplastic contact as the whole analysis can be conveniently carried out within one single approach.

The description of friction is another important issue in engineering problems. If the friction force is known a priori the modeling of friction results in a variational inequality involving a nonlinear term. An inequality of such structure is often referred to as of second kind, see for example [19]. The approach of Coulomb friction relates the friction force and the normal force with the help of the coefficient of friction. This coefficient depends on the materials of the bodies in contact, cf. [112]. Since for Signorini's problem the normal force and the contact area are not known a priori the existence and uniqueness of a solution is more complicated, cf. [69].

1.1 Notation

In this section, we introduce some notation which will be used throughout this work. For more details on the used Sobolev spaces see [1]. Let $L^2(\Omega, \mathbb{K})$ denote the usual Lebesgue space of squared integrable functions with values in a real vector space $\mathbb{K}$ over a domain $\Omega \subset \mathbb{R}^d$, for $d = 2, 3$. The used choices of $\mathbb{K}$ are for example $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. We write $L^2(\Omega)$ if the choice of $\mathbb{K}$ is obvious from the context. Throughout this work, derivatives are understood in a distributional sense. As usual, the Sobolev space $H^1(\Omega, \mathbb{K})$ is the space of L^2 functions with derivatives in $L^2(\Omega, \mathbb{K}^d)$ and, for positive integers m, the space $H^m(\Omega, \mathbb{K})$ consists of the L^2 functions with derivatives in $H^{m-1}(\Omega, \mathbb{K}^d)$, where we set $H^0(\Omega) := L^2(\Omega)$. As before we write $H^m(\Omega)$ if it is unambiguous. We write $D^j u$ for the j-th derivative of u and set $D^0 u := u$. All these space are Hilbert spaces with the scalar products

$$(u, v)_{m,\Omega} := \sum_{j=0}^{m} \int_\Omega D^j u D^j v.$$

We denote the associated norms by

$$\|u\|_{m,\Omega} := (u, u)_{m,\Omega}^{1/2}.$$

If the notation is unambiguous we omit the set Ω and just write $\|u\|_m = (u, u)_m^{1/2}$ instead. The usual spaces of Sobolev functions on an interval $(0, T)$ with values in $L^2(\Omega)$ is denoted by $H^1(0, T; L^2(\Omega, \mathbb{R}^m))$.

For $s = m+\theta$ and $1 > \theta > 0$ the space $H^s(\Omega)$ is defined as the interpolation space $H^s(\Omega) := [H^m(\Omega), H^{m+1}(\Omega)]$ in the usual way [76]. With the notation $\Gamma := \partial\Omega$ for the boundary of the set Ω, the trace operator $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ is well defined. We set

$$H_D^m(\Omega) := \{v \in H^m(\Omega) \mid \gamma(v)_{|\Gamma_D} = 0\}$$

for relatively closed $\Gamma_D \subset \Gamma$. Moreover, for $\Gamma_C \subset \Gamma$, $\gamma_C : H^1(\Omega) \ni v \mapsto \gamma(v)_{|\Gamma_C} \in H^{1/2}(\Gamma_C)$ is surjective, if $\bar{\Gamma}_C \subset \Gamma \setminus \Gamma_D$, cf. [69, Theorem 5.6]. With the help of this subset condition it will be possible to avoid the use of more complicated spaces for contact problems. For more details on this topic, we refer to [69]. Let n denote the outer unit normal and let $t = (t_1, t_2) \in \mathbb{R}^{d\times d-1}$ be such that $t_1 \ldots, t_{d-1}$ is a orthonormal basis of the tangential space of Γ_C. We set $\gamma_n(v) := n^\top \gamma_C(v) \in \mathbb{R}$ and $\gamma_t(v) := t^\top \gamma_C(v) \in \mathbb{R}^{d-1}$ and note that $\gamma = (\gamma_n, \gamma_t)$ if for Γ the normal and tagential spaces are well defined. For $v \in H^1(\Omega, \mathbb{R}^d)$ and $\tau \in H^1(\Omega, \mathbb{R}^{d\times d})$, we may sometimes use of the abbreviated form $v_n = n^\top \gamma_C(v)$, $\tau_n = n^\top \gamma_C(\tau)$, $\tau_{nt} = t^\top(n^\top \gamma_C(\tau))$, etc.

Throughout this work, we identify $L^2(\Omega)$ with its dual spaces and write $\tilde{H}^{-s}(\Omega)$ for the dual space of $H^s(\Omega)$ and $s \geq 0$ with the norm $\|\mu\|_{-s} := \sup_{v \in H^s(\Omega), \|v\|_s = 1} \langle \mu, v \rangle$. For a product space $H = H^{-s} \times H^{-s}$, we use the norm $\|(\mu_1, \mu_2)\|_{-s} := \|(\mu_1)\|_{-s} + \|(\mu_2)\|_{-s}$. Furthermore, we note that $L^2(\Omega) \subset \tilde{H}^{-s}(\Omega)$. For a general Banach space X, we denote its dual space by X'. The duality pairing of two elements $x \in X$ and $x' \in X'$ reads $\langle x, x' \rangle$.

For $A, B \in \mathbb{R}^{d \times d}$, we set $A : B := \sum_{i,j=1}^{d} A_{ij} B_{ij}$ and $|A| = (A : A)^{1/2}$. We write $\mathrm{tr}(A) := \sum_{j=1}^{d} A_{jj}$ for the trace of a Matrix A. Moreover, $\mathbb{R}^{d \times d}_{\mathrm{sym}}$ denotes the space of symmetric square matrices.

1.2 Linear Elasticity

For a better understanding of elastoplasticity we briefly introduce the well known linear elastic material law. For a more detailed introduction see, for example, [78, 21]. The formulation describes the deformation of a body $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We assume its boundary Γ to be sufficiently smooth and to consist of a closed part $\Gamma_D \subset \Gamma$ of positive measure and a possibly empty part $\Gamma_N := \Gamma \setminus \Gamma_D$. The deformation of the body is given by a function $u \in V := H^1_D(\Omega, \mathbb{R}^d)$, whereas the exterior volume force and surface traction acting on the body are described by functions $f_\Omega \in L^2(\Omega, \mathbb{R}^d)$ and $f_N \in L^2(\Gamma_N, \mathbb{R}^d)$, respectively. Hooke's law links the linearized Green strain tensor $\varepsilon(u) := (\nabla u + (\nabla u)^\top)/2$ and the stress tensor σ via the elasticity tensor $\mathbb{C}$, i.e., $\sigma(u) := \mathbb{C}\varepsilon(u)$. The tensor $\mathbb{C}$ is assumed to be symmetric, i.e. $\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jilk}$, and uniformly elliptic, i.e. for all $\tau \in \mathbb{R}^{d \times d}_{\mathrm{sym}}$ there holds $\mathbb{C}\tau : \tau \geq \kappa \tau : \tau$ with a constant $\kappa > 0$.

Furthermore, we set $\sigma_n(u) := n^\top \sigma(u)$ with the outer unit normal n of Γ. The stress $\sigma(u)$ fulfills the equilibrium of interior and exterior force which is given by

$$
\begin{aligned}
\operatorname{div} \sigma(u) + f_\Omega &= 0 \qquad \text{in } \Omega, \\
\sigma_n(u) + f_N &= 0 \qquad \text{on } \Gamma_N.
\end{aligned} \tag{1.1}
$$

The weak formulation of (1.1) is obtained by multiplication with a test function $v \in H^1_D(\Omega)$ and integration by parts over Ω which results in the variational equation

$$
(\sigma(w), \varepsilon(v))_0 = (f_\Omega, v)_0 + (f_N, v)_{0,\Gamma_N}. \tag{1.2}
$$

Under the above assumptions there exists a unique weak solution $u \in H^1_D(\Omega, \mathbb{R}^d)$ of (1.2) due to the Korn's inequality and the Lax-Milgram Lemma.

1.3 Elastoplasticity

The constitutive equations of linear elasticity can easily be extended and modified to model elastoplastic material behavior. In the common approach the laws of thermodynamics give rise to the constitutive equations of elastoplasticity. Here, we only present these constitutive equations without any derivation for details on the thermodynamic consideration we refer to [57, 101]. One principle is to consider so-called internal variables which describe the dissipation of mechanical energy due to plastic deformations. Hence, we start with the introduction of the internal variable α, its conjugated force A and the plastic strain p. Next, we define the generalized stress $\Sigma := (\sigma, A)$ and the generalized plastic strain $P := (p, \alpha)$. The generalized stress is assumed to be restricted to a set of admissible stresses which depends on the chosen model. Throughout this work, we assume admissible stresses to be given by a von Mises yield condition, cf. [26, 57, p.62]. The dissipation functional j is the support function of the set of admissible generalized stresses, cf. [4].

The internal variables α are assumed to be in $H^1(0,T;L^2(\Omega,\mathbb{R}^m))$ and the plastic strain p in $H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ with $tr(p) = 0$. The dimension of the internal variables $m \in \mathbb{N}$ depends on the assumed hardening law. We also allow $m = 0$ which indicates that apart from the plastic strain no other internal variables are present. This is for example the case when modeling linear kinematic hardening behavior or perfect plasticity. Further, let $\dot{f}$ denote the derivative in time of a function f and let ∂f denote the subgradient of convex function f. Moreover, j denotes the plastic dissipation functional. Therewith, the plastic flow law is given as

$$\Sigma \in \partial j(\dot{P}). \tag{1.3}$$

The Fenchel-Legendre dual j^* of j is defined as

$$j^*(\Sigma) := \sup_{\tilde{Q}\in H^1(0,T;L^2(\Omega,\mathbb{R}^m))\times H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} (\Sigma : \tilde{Q} - j(\tilde{Q})),$$

see for example [49, 114]. Hence, the flow law equivalently reads

$$\dot{P} \in \partial j^*(\Sigma). \tag{1.4}$$

Throughout this work, we only consider lower semi-continuous and convex dissipation functionals j which are positively homogeneous of degree one, i.e. $j(cP) = cj(P)$ for $c \geq 0$. Furthermore, Let $\tilde{\mathbb{H}}$ denote the symmetric and positive definite forth order hardening tensor which depends on the properties of the modeled material. We assume the internal variables α to be coupled with A by a hardening law of the form

$$A = \tilde{\mathbb{H}}\alpha. \tag{1.5}$$

The strain is additively split into a plastic p and an elastic ε_e component, i.e., $\varepsilon = \varepsilon_e + p$. As in linear elasticity, the tensor of elasticity couples the stress to the elastic part of the strain

$$\sigma(u, p) := \mathbb{C}(\varepsilon(u) - p). \tag{1.6}$$

The stress σ and the external force f_Ω and traction f_N still fulfill the equilibrium equations (1.1). We consider the exterior forces to be independent of time as we will focus on the spatial variational inequality which arises within one time step of time discrete quasistatic elastoplasticity. This given, we are able to define the primal problem of elastoplsaticity with hardening.

Definition 1.1. *The strong formulation of the primal problem of elastoplasticity with linear hardening consists of finding* (u, P) *such that*

$$\operatorname{div} \sigma(u,p) + f_\Omega = 0 \qquad \textit{in } \Omega, \tag{1.7}$$

$$\sigma_n(u,p) + f_N = 0 \qquad \textit{on } \Gamma_N \tag{1.8}$$

$$\dot{P} \in \partial j^*(\sigma, A) \tag{1.9}$$

$$A = \tilde{\mathbb{H}}\alpha. \tag{1.10}$$

The dissipation functional j and the hardening tensor $\tilde{\mathbb{H}}$ depend on the considered hardening law. We will specify some hardening models below.

Remark 1.2. The name primal problem indicates that there also exists another formulation of the problem. In the so-called dual problem the sought quantities are the displacement u and the generalized stress field $\Sigma = (\sigma, \chi)$. The unknown χ is the conjugated force with respect to all internal variables including the plastic strain. The dual formulation is equivalent to the primal problem but will not be considered in this thesis instead we refer to [57, Chapter 8] for more details.

Before we present the weak form of the primal problem, we first introduce the spaces

$$Q := \{q \in L^2(\Omega; \mathbb{R}^{k\times k}_{\mathrm{sym}}) \mid \operatorname{tr}(q) = 0 \text{ a.e. in } \Omega\}$$

and

$$W := H^1_D \times Q \times L^2(\Omega, \mathbb{R}^m)$$

with the dimension of internal variables $m \in \mathbb{N}$. If $m = 0$, the previous and the following definitions should be understood as if stated without the terms concerning the internal variables. For example, we set $W := H^1_D \times Q$ if $m = 0$, i.e., the plastic strain is the only internal variable. The space W is a Hilbert space with the norm $\|(v, q, \beta)\|^2_W := \|v\|^2_1 + \|q\|^2_0 + \|\beta\|^2_0$.

Further, we define a bilinear form $a : W \times W \to \mathbb{R}$, and functionals $\Psi : W \to \mathbb{R}$ and $\mathcal{F} \in W'$ as

$$\begin{aligned} a(w,z) &:= (\sigma(u,p), \varepsilon(v) - q)_0 + (\tilde{\mathbb{H}}\alpha, \beta)_0, \\ \Psi(z) &:= \int_\Omega j(q,\beta)\, dx \\ \mathcal{F}(z) &:= (f_\Omega, v)_0 + (f_N, v)_{0,\Gamma_N} \end{aligned}$$

with $w := (u, p, \alpha), z := (v, q, \beta) \in W$. Since the functionals Ψ and $\mathcal{F}$ do not really depend on all components we may sometimes write $\Psi(q, \beta)$ or $\mathcal{F}(v)$ instead of $\Psi(z)$ or $\mathcal{F}(z)$, respectively.

We indicate that the bilinear form a is symmetric, continuous and W-elliptic due to the assumptions on $\mathbb{C}$ and $\tilde{\mathbb{H}}$. That is to say, $a(w, z) = a(z, w)$ and there exist constants $\nu_0, \nu_1 > 0$ such that

$$a(z, w) \leq \nu_0 \|w\|_W \|z\|_W, \quad \nu_1 \|w\|_W^2 \leq a(w, w) \tag{1.11}$$

for all $w, z \in W$, see [57, Proof of (7.52)]. Furthermore, the functional Ψ is convex and lower semi-continuous. The weak form of the primal problem follows from the strong form with some integration by parts.

Definition 1.3. *The weak form of the primal problem of elasticity with linear hardening consists of finding $w \in H^1(0, T; W)$ such that the variational inequality*

$$a(w, z - \dot{w}) + \Psi(z) - \Psi(\dot{w}) \geq \mathcal{F}(z - \dot{w}) \tag{1.12}$$

holds for all $z \in H^1(0, T; W)$.

In order to obtain a time discrete version of the weak form let t_i for $i = 0, \dots N$, denote the points of the time discretization. Moreover, we denote by w_i the approximations of $w(t_i)$ and set $\Delta^i w := w_i - w_{i-1}$. Hence, the discretization of the weak form with an implicit Euler scheme results in

$$a(\Delta^i w, z - \Delta^i w) + \Psi(z) - \Psi(\Delta^i w) \geq \mathcal{F}(z - \Delta^i w) - a(w_{i-1}, z - \Delta^i w), \tag{1.13}$$

Here, we used the positive homogeneity of Ψ.

Remark 1.4. We are not interested in the time discretization error and its influence on the spatial discretization error. But we note that a discretization by the Crank-Nicolson method would result in

$$1/2 a(\Delta^i w, z - \Delta^i w) + \Psi(z) - \Psi(\Delta^i w) \geq \mathcal{F}(z - \Delta^i w) - a(w_{i-1}, z - \Delta^i w).$$

For the comparison of the two discretization errors, and the interaction of spatial and time discretization, we refer to [57, 3, 28].

From here on, we will focus on one time step of quasi static time discrete elastoplasticity. Without loss of generality, we only consider the first time step and write w instead of w_1. Furthermore, we assume homogeneous initial conditions, i.e., $u(0) = 0$ and $P(0) = 0$. With those assumptions made, the variational inequality (1.13) results in the following weak formulation of quasi static time discrete elastoplasticity.

Definition 1.5. *Let $w \in W$ fulfill*

$$a(w, z - w) + \Psi(z) - \Psi(w) \geq \mathcal{F}(z - w), \tag{1.14}$$

for all $z \in W$ then w is called the weak solution of quasi static time discrete elastoplasticity.

Due to the properties of a, Ψ, and $\mathcal{F}$ there exists a unique solution $w \in W$ of (1.14) cf. [57, Lemma 7.1]. This solution is also the unique minimizer of the energy functional

$$E_P(z) := H(z) + \Psi(z) \tag{1.15}$$

with $H(z) := 1/2a(z,z) - \mathcal{F}(z)$.

Next, we present some common hardening laws and the corresponding dissipation functionals. In the presence of combined isotropic and linear kinematic hardening behavior the dissipation functional is given as

$$j(q,\beta) := \begin{cases} \sigma_y |q| & \text{if } |q| - \beta \leq 0 \\ \infty & \text{otherwise} \end{cases}$$

with $\sigma_y > 0$ being the initial yield stress and $\beta \in L^2(\Omega;\mathbb{R})$ the accumulated plastic strain. The Hardening law is described by $\mathbb{H}$ the linear kinematic and $H(x) > 0$ the isotropic hardening moduli. Thereby, the bilinear form a reads

$$a(w,z) := (\sigma(u,p), \varepsilon(v) - q)_{L^2(\Omega;\mathbb{R}^{d\times d})} + (\mathbb{H}p, q)_{L^2(\Omega;\mathbb{R}^{d\times d})} + (H\alpha, \beta)_{L^2(\Omega;\mathbb{R})},$$

with $w := (u,p,\alpha), z := (v,q,\beta) \in W := H^1_D(\Omega) \times Q \times L^2(\Omega,\mathbb{R})$. Furthermore, the minimization with respect to the internal variable α and the homogeneous initial condition result in the elimination of α since it can be expressed in terms of p, see [25, Examples 4.5 and 4.6] for details. In this case, the bilinear form reads

$$a(w,z) = (\sigma(u,p), \varepsilon(v) - q)_{L^2(\Omega;\mathbb{R}^{d\times d})} + ((\mathbb{H} + H\sigma_y\mathbb{I})p, q)_{L^2(\Omega;\mathbb{R}^{d\times d})}.$$

Moreover, the dissipation functional is given by

$$j(q) := \sigma_y|q|.$$

Furthermore, if only linear kinematic hardening is modeled the plastic strain p still is the only internal variable present. The bilinear form a reduces to

$$a(w,z) := (\sigma(u,p), \varepsilon(v) - q)_{L^2(\Omega;\mathbb{R}^{d\times d})} + (\mathbb{H}p, q)_{L^2(\Omega;\mathbb{R}^{d\times d})},$$

with the modulus of linear kinematic hardening $\mathbb{H}$. The dissipation functional remains the same as for the combined hardening.

In the following we restrict ourselves to the case of linear kinematic hardening whenever the hardening tensor in a is used explicitly. However, all results obviously still hold if $\mathbb{H}$ is replaced by $\mathbb{H} + H\sigma_y\mathbb{I}$.

Remark 1.6. In order to approximate nonlinear kinematic hardening behavior it is possible to introduce multiple yield surfaces with different modulus of hardening and plastic strain variables, see [38, 65]. However, this does not induce additional mathematical complexity then in the case of a single yield surface and will be included implicitly in the following theoretical analysis.

Remark 1.7. The constant ν_1 in (1.11) crucially depends on the hardening parameter $\mathbb{H}$. This can cause numerical problems for the approximation of the primal variables when the hardening term $\mathbb{H}$ is close to zero in the variational inequality (1.14). Furthermore, when the hardening behavior is completely absent the material model is called perfect elastoplasticity. Without the hardening term $\mathbb{H}p : p$, the bilinear form a is no longer W-elliptic and therefore the unique solvability of the primal problem (1.14) is not given. However, the stresses can still be well approximated, see [27, 109].

Additionally to the energy minimization problem (1.15) and the variational inequality (1.14) we consider a saddle point formulation based on the introduction of a Lagrange formulation. This results in an equivalent problem to the other two formulations of the primal problem of elastoplasticity with linear kinematic hardening. Occasionally, we will call the saddle point problem mixed formulation due to the introduction of additional variables. This is not related to the mixed method of [67, 82, 56] where the term refers to the mixed nature of the dual discretized problem.

The basic idea to establish the saddle point problem is to replace the nonlinear and non differentiable Functional Ψ by a bilinear form. To this end, we introduce the set

$$\Lambda_P := \{\mu \in Q \mid \mu : \mu \leq 1\}.$$

Next, we set

$$\mu_q := \begin{cases} q/|q| & \text{if } |q| \neq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now, from Cauchy's inequality and the fact that $\mu_q \in \Lambda_P$, we observe

$$\Psi(q) = \sup_{\mu \in \Lambda_P} (\mu, \sigma_y q)_0.$$

Moreover, we define the Lagrangian

$$\mathcal{L}_P(z, \mu) := H(z) + (\mu, \sigma_y q)_0.$$

Hence, the energy functional E_P can be rewritten equivalently as

$$E_P(z) = \sup_{\mu \in \Lambda_P} \mathcal{L}_P(z, \mu) = H(z) + \sup_{\mu \in \Lambda_P} (\mu, \sigma_y q)_0.$$

This gives rise to the following saddle point problem.

Definition 1.8. *The saddle point problem in elastoplasticity with linear kinematic hardening consist in finding $(w, \lambda) \in W \times \Lambda_P$ such that*

$$\mathcal{L}_P(w, \mu) \leq \mathcal{L}_P(q, \mu) \leq \mathcal{L}_P(q, \lambda) \tag{1.16}$$

holds for all $(z, \mu) \in W \times \Lambda_P$. A pair (w, λ) fulfilling (1.16) *is called a saddle point of $\mathcal{L}_P$*

A saddle point $(w, \lambda) \in W \times \Lambda_P$ of $\mathcal{L}_P$ can be equivalently characterized by

$$E_P(w) = \inf_{z \in W} E_P(z) = \inf_{z \in W} \sup_{\mu \in \Lambda_P} \mathcal{L}(z, \mu) = \mathcal{L}_P(w, \lambda).$$

Furthermore, another characterization is given by the stationary condition

$$\begin{aligned} a(w,z) + (\lambda_P, \sigma_y q)_0 - \mathcal{F}(z) &= 0 \\ (\mu_P - \lambda_P, \sigma_y p)_0 &\leq 0 \end{aligned} \tag{1.17}$$

which has to hold for all $(z,\mu) \in W \times \Lambda_P$, cf. [97].

Immediately, the question arises whether a saddle point of $\mathcal{L}_p$ exists and if so is it the only one. In [49], Ekeland and Témam give general conditions under which the existence is guaranteed. For the sake of completeness, we state the next theorem in the more general setting of reflexive Banach spaces like in [49]. Though, we only consider the special case of Hilbert spaces throughout the present work.

Theorem 1.9. *Let $\mathcal{L} : Z \times \Lambda \to \mathbb{R}$, where Z is a reflexive Banach space and Λ a closed and convex subset of a possibly different reflexive Banach space. Furthermore, let hold the conditions:*

i. $-\mathcal{L}(z,\cdot)$ is convex and weakly lower semi-continuous for all $z \in W$.

ii. $\mathcal{L}(\cdot,\mu)$ is convex and weakly lower semi-continuous for all $\mu \in \Lambda$.

iii. There exists $\mu \in \Lambda$, such that $\mathcal{L}(\cdot,\mu)$ is coercive.

iv. Λ is bounded or $\mu \mapsto \sup_{z\in W} -(\mathcal{L}(z,\mu))$ is coercive.

Then, there exists a saddle point (w,λ) of $\mathcal{L}(z,\mu)$.

Proof. See [49, Propositions VI.2.1, VI.2.2 and Remark VI.2.1] □

Theorem 1.9 given, we observe that, for the Lagrangian $\mathcal{L}_P$, the condition *i* concerning the Lagrange multiplier reduces to a condition on the L^2 inner product, which obviously is fulfilled. Likewise, conditions *ii* and *iii* directly follow from the properties of the bilinear form a and the L^2-inner product. Finally, Λ_P is bounded in $L^2(\Omega)$ due to the boundedness of Ω. Hence, a saddle point (w,λ_P) exists. The first component is also the minimizer of the functional E_P and consequently unique. Let (w,λ_P) and $(w,\tilde{\lambda}_P)$ be two saddle points of $\mathcal{L}_P$ then

$$\sigma_y \|\lambda_P - \tilde{\lambda}_P\|_0 = \sup_{q\in Q, \|q\|_0 = 1} (\lambda_P - \tilde{\lambda}_P, q)_0 = 0$$

and thus the Lagrange multiplier is also unique. Altogether, we have shown the existence of a unique saddle point

We define the deviatoric part of $\tau \in L^2(\Omega, \mathbb{R}^{d\times d})$ by

$$\operatorname{dev}(\tau) := \tau - \frac{1}{d}\operatorname{tr}(\tau)\mathbb{I}$$

Next, we observe that

$$(\operatorname{tr}(\tau)\mathbb{I}, q)_0 = (\operatorname{tr}(\tau), \operatorname{tr}(q))_0 = 0.$$

for arbitrary $\tau \in L^2(\Omega, \mathbb{R}^{d\times d})$ and all $q \in Q$. Hence,

$$(\tau, q)_0 = (\tau - \mathrm{tr}(\tau)\mathbb{I}, q)_0 = (\mathrm{dev}(\tau), q)_0.$$

From this, it is easy to see that

$$\lambda_P = \mathrm{dev}(\sigma(w) - \mathbb{H}p)/\sigma_y,$$

cf. [97]. At the end of this chapter, we will show this also holds for a model of frictional contact in elastoplsaticity. The quantity $\mathrm{dev}(\sigma(w) - \mathbb{H}p)$ is sometimes referred to as plastic stress, cf. Section 1 in [38].

1.4 Contact

In this section we present a contact problem without friction for elastoplasticity with hardening. We focus on Signorini contact conditions which are well known from the case of linear elastic material behavior described in Section 1.2. For more details on the Signorini problem of linear elasticity, we refer to [69, 64]. The Signorini problem is a model for the contact of an elastic body with a rigid foundation. The contact zone is the part of the boundary of the body which actually is in contact with the rigid object. Usually, this zone is not known a priori. However, in practice, a zone of possible contact can often be determined in advance. Hence in this work, we consider a contact boundary $\Gamma_C \subset \Gamma$ with $\bar{\Gamma}_C \subset \Gamma \setminus \Gamma_D$ and $\Gamma_C \cap \Gamma_N = \emptyset$ which is assumed to contain the actual contact zone. The assumption $\bar{\Gamma}_C \subset \Gamma \setminus \Gamma_D$ assures that a part of boundary with positive surface measure lies between the Dirichlet part Γ_D and the contact part Γ_C. In this way, we can avoid the introduction of $H^1_{0,0}$ spaces, their definition and more details can be found in [69, 64].

In order to impose a non penetration condition on the contact boundary, we have to introduce a concept of distance between the body and the obstacle. Since we only consider small deformations we will use the distance between the initial configuration and the rigid foundation as a reasonable approximate guess. To this end, we set $\bar{x} = (x_1, x_2)$ and $\bar{x} = (x_1)$ for $x = (x_1, x_2, x_3) \in \Gamma \subset \mathbb{R}^3$ and for $x = (x_1, x_2) \in \Gamma \subset \mathbb{R}^2$, respectively. Moreover, we assume that there exist functions $\psi, \phi : \mathbb{R}^{d-1} \to \mathbb{R}$ whose graphs include the contact boundary , i.e., $\Gamma_C \subset \Psi(\mathbb{R}^{d-1})$, and the part of the obstacle's boundary that possibly is in contact, respectively. This given, the so-called linearized gap function

$$g(x) = \frac{\phi(\bar{x}) - \psi(\bar{x})}{(1 + |\nabla\psi(\bar{x})|^2)^{1/2}}$$

represents the initial gap between the body's reference configuration and the obstacle [69, Chapter 2]. Additionally, if ϕ and ψ are sufficiently smooth then $g \in H^{1/2}(\Gamma_C)$. Note that the gap function is allowed to be negative. In fact, this is a very common case since in such a way it is possible to model the stationary limit of the setting where a body hits or is (slowly) hit by an obstacle.

For $z = (v, q) \in W$, we set $\gamma(w) := \gamma(v)$, $\gamma_n := \gamma_n(v)$, $j(z) := j(q)$, etc. Now, we define the strong formulation of the elastoplastic contact problem.

Definition 1.10. *The contact problem for elastoplasticity with hardening in strong form is given as to find $w = (u, p) \in W$ such that*

$$\operatorname{div} \sigma(w) + f_\Omega = 0 \qquad \text{in } \Omega, \tag{1.18}$$

$$\sigma_n(w) + f_N = 0 \qquad \text{on } \Gamma_N, \tag{1.19}$$

$$\sigma(w) - \mathbb{H}p \in \partial j(p) \tag{1.20}$$

$$\gamma_n(w) - g \leq 0, \quad \sigma_{nn} \leq 0, \quad (\gamma_n(w) - g)\sigma_{nn} = 0, \quad \sigma_{nt} = 0 \qquad \text{on } \Gamma_C. \tag{1.21}$$

The first condition in (1.21) is the restriction of the displacement in normal direction such that the body does not penetrate the obstacle. The second one restricts the normal component of the surface tension to point into the body. Next, the complementary condition induces that if there is no contact then the normal stress is zero. Finally, the absence of frictional forces is implied by the last condition.

Definition 1.11. *A function*

$$w \in K := \{z = (v, q) \in W \mid \gamma_n(z) \leq g \text{ a.e. on } \Gamma_C\}$$

is called the weak solution of the contact problem for elastoplasticity with hardening if the variational inequality

$$a(w, z - w)_0 + \Psi(z) - \Psi(w) - \mathcal{F}(w - z) \geq 0, \tag{1.22}$$

holds for all $z \in K$.

The restriction included in the set K is the only difference between the variational problem here and the variational inequality (1.14) without contact conditions. It is derived from the strong form with the help of integration by parts in the same way as in the absence of contact conditions.

In the same way as before, the variational inequality is equivalent to a energy minimization problem.

Definition 1.12. *The energy minimization problem of the contact problem for elastoplasticity with linear hardening consists of finding $w \in K$ such that*

$$E_P(w) = \inf_{z \in K} E_P(z) \tag{1.23}$$

with the functional E_P as defined in (1.15)*.*

The set K is convex and bounded, cf. [69, Theorem 5.7]. Since the energy functional is W-coercive it is also coercive on K and the existence of a minimizer follows from the next theorem.

Theorem 1.13 ([69, Theorem 3.1, p.33]). *Let K be a convex, closed and nonempty subset of a reflexive Banach space W. Moreover, Let $E : K \to \mathbb{R}$ be weakly lower semi-continuous and coercive, then there exists a $w \in K$ that fulfills* (1.23).

The uniqueness of the minimizer is guaranteed if the functional E is additionally strictly convex. In our case, this directly follows from the ellipticity of a and the convexity of j.

We reformulate (1.23) as a saddle point problem of a Lagrange functional and shift the restrictions of the set K into the definition of the space of Lagrange multipliers. To this end, we introduce the sets

$$H_+^{1/2}(\Gamma_C) := \{v \in H^{1/2}(\Gamma_C) \mid v \geq 0 \text{ a.e. on } \Gamma_C\}$$

and

$$\tilde{H}_+^{-1/2}(\Gamma_C) := \{\mu_C \in \tilde{H}^{-1/2}(\Gamma_C) \mid \langle \mu_C, v\rangle \geq 0 \text{ for all } v \in H_+^{1/2}\}.$$

Moreover, the Lagrange functional $\mathcal{L}_C : W \times \tilde{H}_+^{-1/2}(\Gamma_C) \to \mathbb{R}$ reads

$$\mathcal{L}_C(z, \mu_C) := E(z) + \langle \mu_C, \gamma_n(z) - g\rangle. \tag{1.24}$$

Next, we observe that there holds

$$E(w) = \inf_{z \in K} E(z) = \inf_{z \in W} \sup_{\mu_C \in \tilde{H}_+^{-1/2}(\Gamma_C)} \mathcal{L}_C(z, \mu_C) \tag{1.25}$$

by the Hahn-Banach theorem, see [64, Section 1.1.2]. It follows directly that if (w, λ_C) is a saddle point of $\mathcal{L}_C$ then w is the unique minimizer of E. In the same way as above, the existence of a saddle point can be guaranteed by Theorem 1.9.

In order to show this, we set $\Lambda_C := \tilde{H}_+^{-1/2}(\Gamma_C)$. In the same way as in [98], the conditions *i*, *ii* and *iii* of Theorem 1.9 are fulfilled due to the assumptions on H and j, and the Hahn-Banach theorem. Thus, if in addition the condition *iv* holds, the Lagrange functional $\mathcal{L}_C$ has at least one saddle point. Indeed, we will prove that this holds due to the following lemma.

Lemma 1.14. *If there exists $\alpha > 0$ with*

$$\alpha \|\mu_C\|_{-1/2,\Gamma_C} \leq \sup_{z=(v,q) \in W, \|z\|_W = 1} \langle \mu_C, \gamma_n(z)\rangle, \tag{1.26}$$

then the mapping

$$\mu_C \mapsto \sup_{z \in W} -(E(z) + \langle \mu_C, \gamma_n(z) - g\rangle = \sup_{z=(v,q) \in W, \|z\|_W = 1} -\mathcal{L}(z, \mu_C)$$

is coercive.

Proof. From the properties of $\mu \in \Lambda_C$, we know that there exists a unique w_μ such that

$$a(w_\mu, \tilde{z} - w_\mu) + \Psi(\tilde{z}) - \Psi(w_\mu) \geq \mathcal{F}(\tilde{z} - w_\mu) - \langle \mu, \gamma_n(\tilde{z}) - \gamma_n(w_\mu)\rangle \tag{1.27}$$

holds for all $\tilde{z} \in W$. We choose $\tilde{z} := (z + w_\mu) \in W$ with arbitrary $z \in W$ and observe

$$\begin{aligned}\langle \mu, \gamma_n(z + w_\mu) - \gamma_n(w_\mu)\rangle &\leq a(w_\mu, z) + \Psi(z + w_\mu) - \Psi(w_\mu) - \mathcal{F}(z)\\ &\leq a(w_\mu, z) + \Psi(z) + \Psi(w_\mu) - \Psi(w_\mu) - \mathcal{F}(z)\\ &\leq \nu_0\|w_\mu\|_W\|z\|_W + \sigma_y^2|\Omega|\|z\|_W + \|\mathcal{F}\|_{W'}\|z\|_W,\end{aligned}$$

where we used the continuity of a and $\mathcal{F}$ and $\Psi(z) = \int_\Omega \sigma_y(q : q)^{(1/2)} \leq \sigma_y|\Omega|^{1/2}\|q\|_0 \leq \sigma_y|\Omega|^{1/2}\|z\|_W$ which holds due to Hölder's inequality. Thus, we have

$$\begin{aligned}\alpha\|\mu\|_{-1/2,\Gamma_C} &\leq \sup_{z\in W, \|z\|_1 = 1} \langle \mu, \gamma_n(z)\rangle\\ &\leq \nu_0\|w_\mu\|_W + \sigma_y|\Omega|^{1/2} + \|\mathcal{F}\|_{W'},\end{aligned}$$

which implies $\|w_\mu\|_W \to \infty$ as $\|\mu\|_{-1/2,\Gamma_C} \to \infty$. Next, we choose $z = 0$ in (1.27) which gives

$$a(w_\mu, w_\mu) \leq \mathcal{F}(w_\mu) - \Psi(w_\mu) - \langle \mu, \gamma_n(w_\mu)\rangle.$$

Thus, with the ellipticity of a we conclude

$$\begin{aligned}\sup_{z\in W} -\mathcal{L}_C(\mu, z) &\geq -(\frac{1}{2}a(w_\mu, w_\mu) + \Psi(w_\mu) - \mathcal{F}(w_\mu) + \langle \mu, \gamma_n(w_\mu) - g\rangle)\\ &\geq -\langle \mu, -g\rangle + \frac{1}{2}a(w_\mu, w_\mu)\\ &\geq \frac{1}{2}\nu_1\|w_\mu\|_W^2 - \|\mu\|_{-1/2,\Gamma_C}\|g\|_{1/2,\Gamma_C}\\ &\geq \frac{1}{2}\nu_1\|w_\mu\|_W^2 - \|g\|_{1/2,\Gamma_C}\alpha^{-1}(\nu_0\|w_\mu\|_W + \sigma_y|\Omega|^{1/2} + \|\mathcal{F}\|_{W'}).\end{aligned}$$

The assumption follows since $x^2 - cx + b \to \infty$ for $x \to \infty$ and constants $c, b \in \mathbb{R}$. $\square$

Before we finally proof the existence of a unique saddle point we present a stationary condition which is similar to the one in Section 1.3. We briefly show that it is equivalent to the saddle point formulation of $\mathcal{L}_C$. This stationary condition will be useful to establish the uniqueness of a saddle point.

Lemma 1.15. *The pair (w, λ_C) is a saddle point of $\mathcal{L}_C$ if and only if for all (z, μ_C) it satisfies the stationary conditions*

$$a(w, z - w) + \Psi(z) - \Psi(w) + \langle \lambda_C, \gamma_n(z - w)\rangle - \mathcal{F}(z - w) \geq 0 \tag{1.28}$$

$$\langle \mu_C - \lambda_C, \gamma_n(w) - g\rangle \leq 0. \tag{1.29}$$

Proof. If and only if $\langle \mu_C, \gamma_n(w) - g\rangle \leq \langle \lambda_C, \gamma_n(w) - g\rangle$ holds for all $\mu_C \in H_+^{-1/2}(\Gamma_C)$, then $\mathcal{L}_C(w,\mu_C) \le \mathcal{L}_C(w,\lambda_C)$. Furthermore, let $\tilde{H}(z) := H(z) + \langle \lambda_C, \gamma_n(z) - g\rangle$ then $\mathcal{L}_C(z,\lambda_C) = (\tilde{H} + \Psi)(z)$ has the same structure as $H + \Psi$. Hence, by the same arguments as in the case of elastoplasticity without contact $\mathcal{L}_C(w,\lambda_C) \le \mathcal{L}_C(z,\lambda_C)$ if and only if (1.28) holds. □

Now, we are able to easily conclude the existence and uniqueness of a saddle point of the mixed formulation of the contact problem for elastoplasticity with linear kinematic and/or isotropic hardening.

Theorem 1.16. *There exists a unique saddle point of $\mathcal{L}_C$.*

Proof. The existence is guaranteed by Theorem 1.9 and Lemma 1.14 since (1.26) holds due to the surjectivity of the trace operator and the closed range theorem, see for instance [87, Lemma I.8]. Let now (w,λ_0) and (w,λ_1) be two saddle points of (1.24). We then choose test functions $z_0 = (v,0) + w$ for λ_0 and $z_1 = (-v,0) + w$ for λ_1 in (1.28) with arbitrary $v \in H^1_D$ and add the resulting inequalities which implies $\langle \lambda_0 - \lambda_1, \gamma_n(v,0)\rangle \ge 0$. Moreover, we have $\langle \lambda_1 - \lambda_0, \gamma_n(v,0)\rangle \ge 0$ if we change the sign of v. Therefore, $\langle \lambda_1 - \lambda_0, \gamma_n(z)\rangle = 0$ for all $z = (v,0)$. The surjectivity of $\gamma_{n|H^1_D \times \{0\}}$ and the definition of the dual norm yields $\lambda_1 = \lambda_0$. □

Remark 1.17. The case that the body is assumed to be in contact with another deformable object instead of a rigid one is known as two body contact problem, cf. [69]. As done for the linear elastic Signorini problem in [99, 70], the arguments used throughout this work can be easily transferred to two body contact problems, cf. [113]

1.5 Frictional Contact

There exist several different approaches to model friction, for some examples we refer to [69]. But in this work, we focus only on the case in which the friction resistance is given as a positive function f_F. If $f_F = 0$ the resulting problem coincides with the frictionless contact problem presented above. This is sometimes referred to as Tresca friction and can be viewed as a special case of Coulomb friction. In Coulomb's law of friction the resistance can be determined by the product of the normal stress and the coefficient of friction, i.e., $f_F = \mu_f|\sigma_{nn}|$. Moreover, the coefficient of friction μ_f only depends on the material and roughness of the surfaces in contact.

The model of Tresca friction can be used to approximate Coulomb friction via an iteration. In such an iterative scheme the frictional resistance is updated by $f_F^j = \mu_F|\sigma_{nn}^{j-1}|$ with σ_{nn}^{j-1} computed from the problem with f_F^{j-1}, for more details see e.g. [48]. To show that this iteration describes an fixed point method it remains to show some contraction property.

In Tresca friction, the given friction resistance restricts the surface tension in tangential direction, i.e, $|\sigma_{nt}| \le f_F$. Furthermore, displacements occur only if $|\sigma_{nt}| = f_F$ and their direction is opposite to the direction of surface tensions.

Definition 1.18. *The strong form of the frictional contact problem in elastoplasticity with hardening is given by*

$$\operatorname{div}\sigma(u,p) + f_\Omega = 0 \qquad \textit{in } \Omega, \tag{1.30}$$
$$\sigma_n(u,p) + f_N = 0 \qquad \textit{on } \Gamma_N, \tag{1.31}$$
$$\sigma(u,p) - \mathbb{H}p \in \partial j(p) \qquad \textit{in } \Omega, \tag{1.32}$$
$$u_n - g \leq 0, \quad \sigma_{nn} \leq 0, \quad (u_n - g)\sigma_{nn} = 0 \qquad \textit{on } \Gamma_C, \tag{1.33}$$
$$|\sigma_{nt}| \leq f_F, \quad (|\sigma_{nt}| - f_F)|u_t| = 0, \quad \exists \zeta \geq 0 : u_t = -\zeta\sigma_{nt} \qquad \textit{on } \Gamma_C. \tag{1.34}$$

As for the frictionless contact problem, we can derive a weak form whose solution is equivalent to the strong solution if only it is sufficiently smooth. We note that there holds $\sigma_{nt}^\top u_t = f_F|u_t|$ on the support of frictional resistance, cf. [87, Lemma II.2], and set

$$\Psi_F(z) := \int_{\Gamma_C} f_F|\gamma_t(z)|.$$

From this, the weak form follows in the usual way from some integration by parts.

Definition 1.19. *Let $w \in K$ be such that the variational inequality*

$$a(w, z-w)_0 + \Psi(z) - \Psi(w) + \Psi_F(z) - \Psi_F(w) - \mathcal{F}(w-z) \geq 0, \tag{1.35}$$

holds for all $z \in K$ then w is called the weak solution of the frictional contact problem in elastoplasticity with linear hardening.

Similar to the problems of the previous sections, we introduce a corresponding energy functional

$$E_F(z) := E_P(z) + \Psi_F(z). \tag{1.36}$$

Like for the frictionless contact problem, this defines the energy minimization problem of finding $w \in K$ such that

$$E_F(w) = \inf_{z \in K} E_F(z). \tag{1.37}$$

And this is again equivalent to the variational inequality. Since E_F fulfills the properties of Theorem 1.13 there exists a minimizer $w \in K$. Additionally, the minimizer is unique due to the strict convexity of E_F which in turn is a direct consequence of the strict convexity of E_P and Ψ_F.

In the same way as the nonlinear plastic dissipation functional in Section 1.3, the restriction due to friction can be treated by Lagrange multipliers. In such a way, the nonlinearity is transferred to a restriction on the set of multipliers. In contrast to the constant yield stress σ_y we allow the frictional resistance function f_F to be zero on the contact boundary. Therefore, we have to define the set of Lagrange multipliers carefully. In principle, we can choose to include f_F in the set of multipliers or in the bilinear form that replaces the nonlinear functional.

In order to include both choices in one notation, let $\tilde{f}_F \in \{1, f_F\}$ and set

$$\zeta(f_F, \tilde{f}_F) := \begin{cases} f_F/\tilde{f}_F & \text{on } \operatorname{supp} \tilde{f}_F, \\ 0 & \text{elsewhere.} \end{cases}$$

We observe that

$$\Psi_F(z) = \sup_{\mu_F \in \Lambda_F} (\mu_F, \tilde{f}_F \gamma_t(z))_{0,\Gamma_C}$$

with

$$\Lambda_F := \{\mu_F \in L^2(\Gamma_C; \mathbb{R}^{d-1}) \mid \mu_F^\top \mu_F \leq \zeta(f_F, \tilde{f}_F)\}.$$

Thus, the energy minimization problem (1.37) can be equivalently rewritten as

$$E_F(w) = \inf_{z \in K} E_F(z) = \inf_{z \in W} \sup_{\mu_C \in \Lambda_C} \sup_{\mu_F \in \Lambda_F} \mathcal{L}_{CF}(z, \mu_C, \mu_F)$$

with the Lagrange functional

$$\mathcal{L}_{CF}(z, \mu_C, \mu_F) := \mathcal{L}_C(z, \mu_C) + (\mu_F, \tilde{f}_F \gamma_t(z))_0.$$

Again, the Lagrange functional defines a saddle point problem.

Definition 1.20. *The triple $(w, \lambda_C, \lambda_F)$ is a saddle point of $\mathcal{L}_{CF}$ if*

$$\mathcal{L}_{CF}(w, \mu_C, \mu_F) \leq \mathcal{L}_{CF}(z, \mu_C, \mu_F) \leq \mathcal{L}_{CF}(z, \lambda_C, \lambda_F) \tag{1.38}$$

holds for all $(z, \mu_C, \mu_F) \in W \times \Lambda_C \times \Lambda_F$.

The existence of a saddle point of $\mathcal{L}_{CF}$ directly follows from the boundedness of Λ_F, Lemma 1.14 and Theorem 1.9. Furthermore, it is unique since $\gamma_t(\ker \gamma_n)$ is dense in $L^2(\Gamma_C; \mathbb{R}^{d-1})$ and $\mu_F = 0$ on $\Gamma_C \setminus \operatorname{supp} f_F$ for all $\mu_F \in \Lambda_F$.

We set $\Lambda := \Lambda_C \times \Lambda_F$ and $\lambda := (\lambda_C, \lambda_F), \mu := (\mu_C, \mu_F) \in \Lambda$ and abbreviate γ_C by γ. Moreover, we introduce the bilinear form

$$b(\mu, \gamma(z) - \tilde{g}) := \langle \mu_C, \gamma_n(z) - g \rangle + (\mu_F, \tilde{f}_F \gamma_t(z))_{0,\Gamma_C}$$

for $\mu \in \Lambda$ and $\gamma(z), \tilde{g} \in (H^{1/2}(\Gamma_C))^d$ with $\tilde{g} = (g, 0)$.

An equivalent characterization of a saddle point is given by the stationary condition.

Theorem 1.21. *If and only if the stationary condition*

$$\begin{aligned} a(w, z - w) + \Psi(z) - \Psi(w) + b(\lambda, \gamma(z - w)) - \mathcal{F}(z - w) &\geq 0 \\ b(\mu - \lambda, \gamma(w) - \tilde{g}) &\leq 0 \end{aligned} \tag{1.39}$$

holds for all $z \in W$ and $\mu \in \Lambda$ then (w, λ) is a saddle point.

Proof. This theorem follows from analog conclusions as in the proof of Lemma 1.15. □

Moreover, we observe that the solution of the saddle point problem fulfills an variational equality and a variational inequality

Lemma 1.22. *Let* $(w, \lambda) \in W \times \Lambda$ *be the solution of the saddle point problem* (1.38). *There holds*

$$a(w, (v, 0)) + b(\lambda, \gamma((v, 0))) - \mathcal{F}((v, 0)) = 0 \tag{1.40}$$
$$a(w, (0, p - q)) \leq \Psi(q) - \Psi(p) \tag{1.41}$$

for all $(v, 0), (0, q) \in W$.

Proof. The assertion follows directly from the first inequality in the stationary condition (1.39) by the insertion of test functions $(v, 0) + w$, $(-v, 0) + w$ and (u, q). □

Equation (1.40) and inequality (1.41) are similar to the ones usually occurring in elastoplasticity with hardening and without contact or friction, as seen for example in [57]. Additionally, we observe that the equation in (1.40) resembles the variational equality of the mixed formulation of the linear elastic Signorini problem in [64].

This observations give rise to the following conclusions concerning the relation of λ and $\sigma(w)$. Let the solution w be sufficiently smooth and choose test functions $z = (v, 0)$, $v \in H^1_D$ in equation (1.40) then

$$\begin{aligned} 0 &= a(w, z) + \langle \lambda_C, \gamma_n(z) \rangle + (\lambda_F, \tilde{f}_F \gamma_t(z))_{0,\Gamma_C} - \mathcal{F}(z) \\ &= (\sigma(u, p), \varepsilon(v))_0 + \langle \lambda_C, \gamma_n(z) \rangle + (\lambda_F, \tilde{f}_F \gamma_t(z))_{0,\Gamma_C} - \mathcal{F}(z) \\ &= (\sigma(u, p)_n, v)_{0,\Gamma_C} + \langle \lambda_C, \gamma_n(z) \rangle + (\lambda_F, \tilde{f}_F \gamma_t(z))_{0,\Gamma_C}. \end{aligned}$$

With test functions $v_n = n^\top v n$ and $v_t = t^\top v t$ we conclude

$$\lambda_C = -\sigma_{nn}, \tag{1.42}$$
$$\tilde{f}_F \lambda_F = -\sigma_{nt}. \tag{1.43}$$

This means that the Lagrange multipliers can be interpreted as normal and tangential stress on the contact boundary, respectively, and therefore have a physical meaning.

We now turn to the mixed formulation where additionally the nonlinearity Ψ is treated with Lagrange multipliers. In this setting, the stationary condition of the Lagrangian includes a third multiplier and the first inequality becomes an equality. We set

$$\Lambda_P := \{q \in Q \mid q : q \leq 1 \text{ a.e. in } \Omega\}.$$

This results in the problem of finding a saddle point of the Lagrange functional

$$\mathcal{L}(z, \mu_C, \mu_F, \mu_P) := \mathcal{L}_{CF}(z, \mu_C, \mu_F) + (\mu_P, \sigma_y q)_{0,\Omega}. \tag{1.44}$$

with $(z, \mu\mu_p) \in W \times \Lambda \times \Lambda_P$. Again, this saddle point problem is equivalent to a stationary condition.

Theorem 1.23. *If and only if* $(w, \lambda, \lambda_P) \in W \times \Lambda \times \Lambda_P$ *is such that*

$$\begin{aligned} a(w,z) + (\lambda_P, \sigma_y q)_0 + b(\lambda, \gamma(z)) - \mathcal{F}(z) &= 0 \\ (\mu_P - \lambda_P, \sigma_y p)_0 + b(\mu - \lambda, \gamma(w) - \tilde{g}) &\leq 0 \end{aligned} \tag{1.45}$$

holds for all $(z, \mu, \mu_P) \in W \times \Lambda \times \Lambda_P$ *then* (w, λ, λ_P) *is a saddle point of* $\mathcal{L}_{CF}$*.*

Proof. This holds by analog conclusions as for Lemma 1.15. □

The existence of λ_P is guaranteed since Λ_P is bounded. Moreover, the same way as in [97], the uniqueness follows from the equation in (1.45), the uniqueness of the other multipliers and

$$\|\lambda_P\|_0 = \sup_{\substack{z=(v,q)\in W,\\ \|z\|_0=1}} (\lambda_P, q)_0.$$

The unique existence of the other components of the saddle point follows just as for the previous formulations.

We test the equation in the stationary condition (1.45) with $z = (0, q)$, $q \in Q$ and conclude

$$\begin{aligned} 0 &= -a(w,z) - (\lambda_P, \sigma_y q)_0 \\ &= (\sigma(w) - \mathbb{H}p, q)_0 - (\lambda_P, \sigma_y q)_0 \\ &= (\operatorname{dev}(\sigma(w) - \mathbb{H}p) - \sigma_y \lambda_P, q)_0 \end{aligned}$$

Since $\operatorname{dev}(\sigma(w) - \mathbb{H}p) - \sigma_y \lambda \in Q$, we obtain

$$\lambda_P = \operatorname{dev}(\sigma(w) - \mathbb{H}p)/\sigma_y. \tag{1.46}$$

As already mentioned at the end of Section 1.3, the quantity $\operatorname{dev}(\sigma(w) - \mathbb{H}p)$ is sometimes referred to as the plastic stress.

Remark 1.24. The stationary conditions presented in this chapter can be interpreted as an iterative step within an Uzawa type algorithm in infinite dimensional spaces, see [52, p. 54]. Although such algorithms would not be preferable for fast computations of numerical solutions, we will see that the saddle point formulations yield a new discrete approximation scheme which could be solved by various approaches.

Remark 1.25. We note that also the yield stress can be included in the definition of the set of Lagrange multipliers in two ways. Moreover, the yield stress can also be considered to be a function. For the ease of notation, we do not consider these settings here, since they does not introduce any further complexity in comparison to the treatment of the friction multiplier. However, it is easy to see that the arguments, that we will use for the two different inclusions of the frictional resistance, still hold in the same way for the yield stress.

2 Discretizations

Just as for variational equalities, the analytic solutions of most variational inequalities as well as the analytic saddle points of the mixed formulation can not be determined directly. Therefore, the question of finding an appropriate numerical approximation is of great importance. The Ritz-Galerkin method is a very common way of producing such a sequence of approximations, see for example [5]. This method is characterized by the usage of discrete spaces W_ℓ instead of the infinite dimensional ones in the original problem. This substitution then results in discrete problems on the spaces $W_\ell = V_\ell \times Q_\ell$. The subscript $\ell \in \mathbb{N}$ is supposed to tend to infinity and in this way indicates that the analytic solution w is approximated by a sequence of solutions w_ℓ on the discrete spaces. Finite element spaces are among the most common choices for discretizations, cf. [22, 21], and we will focus on them in Section 2.3. Nevertheless, the results in Sections 2.1 and 2.2 hold for arbitrary discretization spaces.

Important questions arise immediately, whether there exist discrete solutions and if so how well do they approximate the analytic solution. The question of existence and convergences for arbitrary spaces will be addressed in the first and second section, respectively. In particular, we will present the discrete analogon of the inf-sup condition (1.26). The results are used for the analysis of specific finite element spaces in Section 2.3. Results on the rates of convergence can be found in Chapter 3.

In the following, we focus on elastoplasticity without contact conditions and with frictional contact conditions, respectively. Nevertheless, the results would also apply to the setting where no friction is present.

2.1 Discrete Formulation

Let $V_\ell \subset H^1_D(\Omega)$, $Q_\ell \subset Q$ and $W_\ell := V_\ell \times Q_\ell \subset W$, $\ell = 0, 1, \ldots$, be sequences of finite dimensional spaces. Further, for $\mathscr{L} = 0, 1, \ldots$, let $M_\mathscr{L} \subset L^2(\Gamma_C)$ be a finite dimensional subspace and $\Lambda_{C,\mathscr{L}} \subset M_\mathscr{L} \subset H^{-1/2}(\Gamma_C)$, $\Lambda_{F,\mathscr{L}} \subset M_\mathscr{L}^{d-1} \subset L^2(\Gamma_C; \mathbb{R}^{d-1})$, as well as $\Lambda_{P,\ell} \subset Q_\ell$ be closed and convex. Additionally, we consider $\Lambda_{F,\mathscr{L}}$ and $\Lambda_{P,\ell}$ to be bounded. The sets $\Lambda_{C,\mathscr{L}}$, $\Lambda_{F,\mathscr{L}}$ and $\Lambda_{P,\ell}$ contain the discrete Lagrange multipliers. Nevertheless, they are not assumed to be subsets of the sets Λ_C, Λ_F and Λ_P, respectively. This is usually referred to as a non conform approach.

Note, the choice of different subscripts ℓ and $\mathscr{L}$ does indicate that in general the spaces $M_\mathscr{L}$ need not to be related to W_ℓ in the sense of $M_\mathscr{L} = \gamma(W_\ell)$. For example, the finite element space for the discrete Lagrange multipliers can be defined with polynomial degrees and mesh on the boundary part Γ_C which do not have to coincide with the ones used for W_ℓ. This will be needed in order to construct space such that the discrete inf-sup condition holds. However, if not stated otherwise we assume $\ell = \mathscr{L}$, i.e. W_ℓ and $M_{\mathscr{L}}$ are both on the same level of the discretization loop.

2.1.1 Elastoplasticity

The discrete variational inequality of elastoplasticity with linear kinematic hardening directly arises from the exchange of the infinite dimensional space W by a discrete one W_ℓ.

Definition 2.1. *The solution w_ℓ of the discrete variational inequality for elastoplasticity with hardening is a function in W_ℓ such that*

$$a(w_\ell, z_\ell - w_\ell) + \Psi(z_\ell) - \Psi(w_\ell) \geq \mathcal{F}(z_\ell - w_\ell) \tag{2.1}$$

holds for all $z_\ell \in W_\ell$.

The same arguments as the ones used in the infinite dimensional setting of Section 1.3 show that the solution of the discrete variational inequality is also the unique minimizer of the energy functional E_P over the space W_ℓ ,i.e.,

$$E_P(w_\ell) = \inf_{z_\ell \in W_\ell} E_P(z_\ell).$$

If we want to pass on to a discrete saddle point formulation we have to use a suitable discretization of the set of Lagrange multipliers Λ_P. The choice of the discrete set of Lagrange multipliers $\Lambda_{P,\ell} := \Lambda_P \cap Q_\ell$ would guarantee that the first component of a saddle point $(w_\ell, \lambda_{\mathscr{L}})$ fulfills the variational inequality of Definition 2.1. However, in the light of finite elements, the construction of such a set $\Lambda_{P,\ell}$ for polynomials of degrees greater than one is not clear. As we will see later on, it is much easier to demand the condition $\mu_{P,\ell} : \mu_{P,\ell} \leq 1$ to hold only on a finite set of points per cell. For polynomials of degrees greater than one, this implies a non-conformity in the sense that $\Lambda_{P,\ell} \not\subseteq \Lambda_P$. Therefore, the first component of a discrete saddle point $(w_\ell, \lambda_{P,\ell})$ does not solve the discrete variational inequality nor the energy minimization problem over W_ℓ. Hence, the discrete saddle point problem defined in such a nonconforming way somehow yields a new discretization scheme. However, we will show the convergence of the discrete saddle points to the analytic solution.

Let the set $\Lambda_{P,\ell} \subset Q_\ell$ be convex, closed and bounded. We replace the nonlinear dissipation functional Ψ by a functional Ψ_ℓ with the help of discrete Lagrange multipliers

$$\Psi_\ell(z_\ell) = \sup_{\mu_{P,\ell} \in \Lambda_{P,\ell}} (\mu_{P,\ell}, \sigma_y z_\ell)_0.$$

The existence and the uniqueness of a saddle point $(w_\ell, \lambda_{P,\ell})$ of the Lagrangian $\mathcal{L}_P$ follows as in [97].

Assume that the condition $\lambda_{P,\ell} : \lambda_{P,\ell} \leq 1$ holds on the nodes of a Gaussian quadrature rule. We then have

$$(\lambda_{P,\ell} : p_\ell)(\xi) = |p_\ell|(\xi)$$

for all quadrature nodes ξ. Hence, if the quadrature is exact for $\lambda_\ell : p_\ell$ then this approach results in the same discretization as if the integral in the functional Ψ is approximated by the numerical integration rule. If additionally $\Lambda_{P,\ell} \subset \Lambda_P$ holds we can conclude

$$(\lambda_{P,\ell} : p_\ell)_0 = \Psi(w_\ell)$$

as in [97]. Moreover, if $\Lambda_{P,\ell} \subset \Lambda_P$ we observe that

$$(\lambda_{P,\ell}, \chi(w_\ell))_0 = \sup_{\mu_{P,\ell} \in \Lambda_{P,\ell}} (\mu_{P,\ell}, \chi(w_\ell))_0 = \Psi(w_\ell) = \sup_{\mu_P \in \Lambda_P} (\mu_P, \chi(w_\ell))_0 \geq (\lambda_P, \chi(w_\ell))_0 \tag{2.2}$$

for all $w_\ell \in W_\ell$. In this case, it holds especially

$$\Psi = \Psi_\ell.$$

It is easy to see that the saddle point $(w_\ell, \lambda_{P,\ell})$ of $\mathcal{L}_P$ over $W_\ell \times \Lambda_{P,\ell}$ is the unique point to fulfill the following stationary condition, c.f. [97].

Definition 2.2. *A point $(w_\ell, \lambda_{P,\ell}) \in W_\ell \times \Lambda_{P,\ell}$ fulfills the stationary condition for $\mathcal{L}_P$ if*

$$a(w_\ell, z_\ell) + (\lambda_{P,\ell}, \sigma_y q_\ell)_0 - \mathcal{F}(z_\ell) = 0, \tag{2.3}$$

$$(\mu_{P,\ell} - \lambda_{P,\ell}, \sigma_y p_\ell)_0 \leq 0. \tag{2.4}$$

holds for all $(z_\ell, \mu_{P,\ell}) \in W_\ell \times \Lambda_{P,\ell}$.

The existence of a unique discrete Lagrange multiplier $\lambda_{P,\ell}$ can easily be guaranteed since $\Lambda_{P,\ell}$ is assumed to be closed convex and bounded, and a subset of Q_ℓ, cf. [97].

2.1.2 Frictional Contact

In this section we introduce and discuss discrete versions of the saddle point problems for elastoplasticity with hardening as well as fricitonal and contact conditions. We consider the formulation with two Lagrange multipliers for the contact and the friction condition, respectively, as well as the problem with three Lagrange multipliers. The problem of Section 1.4 where only frictionless contact conditions occur is indirectly included in the frictional contact problem.

We start with the setting (1.38), i.e., the problem for the Lagrangian $\mathcal{L}_{CF}$ with two Lagrange multipliers for contact and friction. Hence, the plastic nonlinearity is still present in the Lagrangian.

Definition 2.3. *The discrete saddle point problem for $\mathcal{L}_{CF}$ consists in finding*

$$(w_\ell, \lambda_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}}) \in W_\ell \times \Lambda_{C,\mathscr{L}} \times \Lambda_{F,\mathscr{L}}$$

such that

$$\mathcal{L}_{CF}(w_\ell, \lambda_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}}) = \inf_{z_\ell \in W_\ell} \sup_{\mu_{C,\mathscr{L}} \in \Lambda_{C,\mathscr{L}}} \sup_{\mu_{F,\mathscr{L}} \in \Lambda_{F,\mathscr{L}}} \mathcal{L}_{CF}(z_\ell, \mu_{C,\mathscr{L}}, \mu_{F,\mathscr{L}}) \tag{2.5}$$

holds for all $(z_\ell, \mu_{C,\mathscr{L}}, \mu_{F,\mathscr{L}}) \in W_\ell \times \Lambda_{C,\mathscr{L}} \times \Lambda_{F,\mathscr{L}}$.

We set

$$\lambda_{\mathscr{L}} := (\lambda_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}})$$

and

$$\Lambda_{\mathscr{L}} := \Lambda_{C,\mathscr{L}} \times \Lambda_{F,\mathscr{L}}.$$

As in the infinite dimensional setting, the saddle point is equivalently characterized by the system of variational inequalities

$$a(w_\ell, z_\ell - w_\ell) + \Psi(z_\ell) - \Psi(w_\ell) + b(\lambda_{\mathscr{L}}, \gamma(z_\ell - w_\ell)) \geq \mathcal{F}(z_\ell - w_\ell) \tag{2.6}$$

$$b(\mu_{\mathscr{L}} - \lambda_{\mathscr{L}}, \gamma(w_\ell) - \tilde{g}) \leq 0. \tag{2.7}$$

This can be shown by the same arguments as in Chapter 1.

Obviously, the result of Lemma 1.22 also holds if the space W, and the sets of multipliers Λ_C and Λ_F are replaced by their discrete counterparts W_ℓ as well as $\Lambda_{C,\mathscr{L}}$ and $\Lambda_{F,\mathscr{L}}$, respectively.

Lemma 2.4. *Let $(w_\ell, \lambda_{\mathscr{L}}) \in W_\ell \times \Lambda_{\mathscr{L}}$ be the solution of* (2.5) *then*

$$a(w_\ell, (v_\ell, 0)) + b(\lambda_{\mathscr{L}}, \gamma((v_\ell, 0))) - \mathcal{F}((v_\ell, 0)) = 0 \tag{2.8}$$

$$a(w_\ell, (0, p_\ell - q_\ell)) + \Psi(p_\ell) - \Psi(q_\ell) \leq 0 \tag{2.9}$$

for all $(v_\ell, 0), (0, q_\ell) \in W_\ell$.

Proof. As in the proof of Lemma 1.22, an appropriate choice of test functions directly yields the assertion. □

Again, as in the previous chapter, we observe that the equation (2.8) resembles the discrete variational equation arising in the discrete mixed formulation of Signorini problems in linear elasticity. As we will see later on, this enables us to apply techniques used by Hlaváček et al. [64] and Schröder [97].

We define the set of admissible discrete displacements

$$K_{\ell,\mathscr{L}} := \{z_\ell \in W_\ell \mid \langle \mu_{C,\mathscr{L}}, \gamma_n(z_\ell) - g \rangle \leq 0 \text{ for all } \mu_{C,\mathscr{L}} \in \Lambda_{C,\mathscr{L}}\}$$

and the discrete nonlinear frictional functional

$$\Psi_{F,\mathscr{L}}(z_\ell) := \sup_{\mu_{F,\mathscr{L}} \in \Lambda_{F,\mathscr{L}}} (\mu_{F,\mathscr{L}}, \tilde{f}_F \gamma_t(z_\ell)).$$

Note, that in general $K_{\ell,\mathscr{L}} \nsubseteq K$ and $\Psi_{F,\ell,\mathscr{L}}(z_\ell) \neq \Psi_F(z_\ell)$ and therefore the discrete displacement is not an admissible function for the previous infinite dimensional energy minimization problems (1.23) and (1.37). However, the first component of a discrete saddle point $(w_\ell, \lambda_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}})$ still is the unique minimizer of an energy functional

$$E_{F,\mathscr{L}}(z_\ell) := E(z_\ell) + \Psi_{F,\mathscr{L}}(z_\ell) \tag{2.10}$$

over $K_{\ell,\mathscr{L}}$. Like in the infinite dimensional case, w_ℓ being a minimizer of (2.10) is equivalent to

$$a(w_\ell, z_\ell - w_\ell) + \Psi(z_\ell) - \Psi(w_\ell) + \Psi_{F,\ell,\mathscr{L}}(z_\ell) - \Psi_{F,\ell,\mathscr{L}}(w_\ell) \geq \mathcal{F}(z_\ell - w_\ell), \tag{2.11}$$

holding for all $z_\ell \in K_{\ell,\mathscr{L}}$. Though, this discrete variational inequality is in general not equivalent to the one obtained by replacing the space W in (1.35) by W_ℓ. Especially, we have $K_{\ell,\mathscr{L}} \neq K \cap W_\ell$ in general.

The existence of the first component of a saddle point $(w_\ell, \lambda_{\mathscr{L}})$ as well as its uniqueness is guaranteed, since it is the unique minimizer of the energy functional $E_{F,\mathscr{L}}$. It remains to find conditions under which the existence and uniqueness of the discrete Lagrange multiplier can be ensured. By some arguments on quotient spaces, it can be shown that if $g \in \gamma_n(W_\ell)$ holds then there exists a saddle point, c.f. [98, Thm. 7]. But in general, the condition $g \in \gamma_n(W_\ell)$ is not fulfilled. However, as for the problems of Chapter 1 this can be overcome if a discrete inf-sup condition holds, cf. [98]. Hence, we have to carefully choose the space $\mathcal{M}_{\mathscr{L}}$ in order to guarantee this condition.

Theorem 2.5. *Let $\alpha > 0$ independent of ℓ such that*

$$\alpha \|\mu_{\mathscr{L}}\|_{-1/2} \leq \sup_{v_\ell \in V_\ell, \|v_\ell\|_V = 1} b(\mu_{\mathscr{L}}, \gamma(z_\ell)) \tag{2.12}$$

holds for all $\mu_{\mathscr{L}} \in M^d_{\mathscr{L}}$, then there exists an unique discrete saddle point $(w_\ell, \lambda_{\mathscr{L}})$.

Proof. In the same way as in the proof of Lemma 1.14 we conclude that (2.12) implies the coercivity of $(\mu_{C,\mathscr{L}}, \mu_{F,\mathscr{L}}) \mapsto \sup_{z_\ell \in W_\ell} -\mathcal{L}_{C,F}(z_\ell, \mu_{C,\mathscr{L}}, \mu_{F,\mathscr{L}})$ which then yields the existence of a saddle point $(w_\ell, \lambda_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}})$. To show the uniqueness of $\lambda_{C,\mathscr{L}}$ and $\lambda_{F,\mathscr{L}}$ let $(w_\ell, \lambda_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}})$ and $(w_\ell, \tilde{\lambda}_{C,\mathscr{L}}, \tilde{\lambda}_{F,\mathscr{L}})$ be two discrete saddle points and $\tilde{z}_\ell \in W_\ell$. Moreover, let $(\tilde{\mu}_{C,\mathscr{L}}, \tilde{\mu}_{F,\mathscr{L}}) := (\lambda_{C,\mathscr{L}} - \tilde{\lambda}_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}} - \tilde{\lambda}_{F,\mathscr{L}})$, $z_\ell := (v_\ell, 0) \in W_\ell$. Thus, (2.8) implies

$$\begin{aligned} \alpha \|\tilde{\mu}_{\mathscr{L}}\|_{-1/2} &\leq \sup_{\bar{v}_\ell \in V_\ell, \|\bar{v}_\ell\|_V = 1} \left(\langle \tilde{\mu}_{C,\mathscr{L}}, \gamma_n(\bar{z}_\ell) \rangle + (\tilde{\mu}_{F,\mathscr{L}}, \tilde{f}_F \gamma_t(\bar{z}_\ell))_{0,\Gamma_C} \right) \\ &\leq 0. \end{aligned}$$

Hence, $(\tilde{\mu}_{C,\mathscr{L}}, \tilde{\mu}_{F,\mathscr{L}}) = (\lambda_{C,\mathscr{L}} - \tilde{\lambda}_{C,\mathscr{L}}, \lambda_{F,\mathscr{L}} - \tilde{\lambda}_{F,\mathscr{L}}) = 0$ since $M^d_{\mathscr{L}}$ is a subspace of $(H^{-1/2}(\Gamma_C))^d$. ☐

Note that we do not demand the norm on the left hand side of the inf-sup condition to be $\|\cdot\|_{H^{-1/2}(\Gamma_C)\times L^2(\Gamma_C)}$ as may have been expected. In Section 2.3, this will turn out to be useful in the proof that the condition holds for the choice of finite element spaces made there.

As in Chapter 1, the inf-sup condition is used to guarantee the uniqueness of the Lagrange multipliers. Furthermore, note that $\Lambda_{F,\mathscr{L}}$ is still closed and bounded and this implies the existence of a discrete multiplier $\lambda_{F,\mathscr{L}}$. However, $\gamma_t(\ker\gamma_n)$ is not necessarily dense in $M_{\mathscr{L}}^{d-1}$ and though the uniqueness of the multiplier is not guaranteed. Therefore, we demand the discrete inf-sup condition to hold for both Lagrange multipliers introduced for contact and friction, respectively.

Next, we consider the case where the full discrete elastoplastic frictional contact problem is reformulated as a discrete saddle point problem using three Lagrange multipliers. Let the set $\Lambda_{P,\ell}\subset Q_\ell$ be convex, closed and bounded.

Definition 2.6. *The discrete saddle point problem for $\mathcal{L}$ consists in finding*

$$(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})\in W_\ell\times\Lambda_{\mathscr{L}}\times\Lambda_{P,\ell}$$

such that

$$\mathcal{L}(w_\ell,\mu_{\mathscr{L}},\mu_{P,\ell})\le\mathcal{L}(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})\le\mathcal{L}(z_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})$$

holds for all $(z_\ell,\mu_{\mathscr{L}},\mu_{P,\ell})\in W_\ell\times\Lambda_{\mathscr{L}}\times\Lambda_{P,\ell}$.

In a similar way as for the other mixed problems, the discrete saddle point $(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})$ is also a stationary point of the Lagrangian $\mathcal{L}$.

Definition 2.7. *The triple $(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})\in W_\ell\times\Lambda_{\mathscr{L}}\times\Lambda_{P,\ell}$ is called a discrete stationary point for $\mathcal{L}$ if and only if*

$$\begin{aligned} a(w_\ell,z_\ell)+(\lambda_{P,\ell},\sigma_y q_\ell)_0+b(\lambda_{\mathscr{L}},\gamma(z_\ell))-\mathcal{F}(z_\ell)&=0\\ (\mu_{P,\ell}-\lambda_{P,\ell},\sigma_y p_\ell)_0+b(\mu_{\mathscr{L}}-\lambda_{\mathscr{L}},\gamma(w_\ell)-\tilde{g})&\le 0 \end{aligned}\tag{2.13}$$

holds for all $(z_\ell,\mu_{\mathscr{L}},\mu_{P,\ell})\in W_\ell\times\Lambda_{\mathscr{L}}\times\Lambda_{P,\ell}$.

Additionally, the first component of the discrete saddle point $(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})$ is the minimizer of the energy functional

$$E_{\ell,\mathscr{L}}(z_\ell)=\sup_{\mu_{\mathscr{L}}\in\Lambda_{\mathscr{L}}}\ \sup_{\mu_{P,\ell}\in\Lambda_{P,\ell}}\mathcal{L}(z_\ell,\mu_{\mathscr{L}},\mu_{P,\ell}).$$

This results in the identity

$$E_{\ell,\mathscr{L}}(w_\ell)=\mathcal{L}(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})=\inf_{z_\ell\in W_\ell}\ \sup_{\mu_{\mathscr{L}}\in\Lambda_{\mathscr{L}}}\ \sup_{\mu_{P,\ell}\in\Lambda_{P,\ell}}\mathcal{L}(z_\ell,\mu_{\mathscr{L}},\mu_{P,\ell}).$$

The existence of unique discrete Lagrange multipliers $\lambda_{\mathscr{L}}$ and $\lambda_{P,\ell}$ can easily be guaranteed since $\Lambda_{P,\ell}$ is assumed to be closed, convex and bounded, and a subset of Q_ℓ, cf. [97].

Remark 2.8. If $\Lambda_{P,\ell} \subset M_{P,\ell} \not\subset Q_\ell$ we extend condition (2.12) to

$$\alpha \|(\mu_{\mathscr{L}}, \mu_{P,\ell})\|_{H^{-1/2}(\Gamma_C)^d \times L^2(\Omega)} \leq \sup_{z_\ell \in W_\ell, \|z_\ell\|_W = 1} \left(b(\mu_{C,\mathscr{L}}, \gamma(z_\ell)) + (\mu_{P,\ell}, \sigma_y(z_\ell))_0 \right) \tag{2.14}$$

and conclude as in Theorem 2.5 that there exists a unique discrete saddle point if (2.14) holds for all $(\mu_{C,\mathscr{L}}, \mu_{F,\mathscr{L}}, \mu_{P,\ell}) \in M_{\mathscr{L}} \times M_{\mathscr{L}}^{d-1} \times M_{P,\ell}$.

2.2 Convergence

In this section, we discuss the convergence of sequences of discrete saddle points of $\mathcal{L}_P$, $\mathcal{L}_{CF}$ or $\mathcal{L}$ to the solution of infinite dimensional problems (1.16), (1.38) or (1.45), respectively. In order to do this, we will not assume any regularity of the analytic solution further than we did in Chapter 1. Whereas the frictional resistance $\tilde{f}_F$ is assumed to be sufficiently regular such that $\|\tilde{f}_F \gamma_t\|$ is bounded. Additionally, we allow for arbitrary series of discrete spaces. The only property needed is the density of their union. Within this section, we will show weak convergence of the sequence of discrete Lagrange multipliers. However, the bounds in the subsequent chapter will ensure strong convergence. If we want to find a priori rates of convergence we will need some assumption on the differentiability of the solution.

2.2.1 Elastoplasticity

The convergence of the direct discretization approach for elastoplasticity under minimal regularity assumptions is found for example in [57, 58]. In the following,but we focus on the convergence of discrete saddle points. If the set of discrete Lagrange multiplier $\Lambda_{P,\ell}$ is a subset of Λ_P a proof of the convergence of a sequence of discrete saddle points can be found in [97] which is based on the ideas in [64]. We just repeat the results without a proof. We will use an approach similar to the conform setting to show the convergence for the nonconforming discretization.

Theorem 2.9 ([97, Theorem 4.5])**.** *Let $\Lambda_{P,\ell} \subset \Lambda_P$. Moreover, assume that there exists a sequence $\{(z_\ell, \mu_{P,\ell})\}$ with $(z_\ell, \mu_{P,\ell}) \in W_\ell \times \Lambda_{P,\ell}$ and $(z_\ell, \mu_{P,\ell}) \to (z, \mu_P)$ as $\ell \to \infty$ for all $(z, \mu_P) \in W \times \Lambda_P$. Then, the sequence $\{w_\ell\}$ strongly converges to w and the sequence of Lagrange multipliers $\{\lambda_{P,\ell}\}$ weakly converges to λ_P as $\ell \to \infty$.*

If $\Lambda_{P,\ell} \nsubseteq \Lambda_P$ but all weak limits μ_P of sequences $\mu_{P,\ell} \rightharpoonup \mu_P$ are in Λ_P, then the convergence follows similar to the arguments of the previous theorem, see also [64, 99] for contact problems in linear elasticity.

Theorem 2.10. *For all sequences $\mu_{P,\ell} \in \Lambda_{P,\ell}$ weakly converging to some μ_P, let hold $\mu_P \in \Lambda_P$. Moreover, assume that there exists a sequence $(z_\ell, \mu_{P,\ell})$ with $(z_\ell, \mu_{P,\ell}) \in W_\ell \times \Lambda_{P,\ell}$ and $(z_\ell, \mu_{P,\ell}) \to (z, \mu_P)$ as $\ell \to \infty$ for all $(z, \mu_P) \in W \times \Lambda_P$. Then, the sequence w_ℓ strongly converges to w and the sequence of Lagrange multipliers $\lambda_{P,\ell}$ weakly converges to λ_P as $\ell \to \infty$.*

Proof. From (2.3), we obtain $\nu_1\|w_\ell\| \leq \|\mathcal{F}\| + \|\lambda_{P,\ell}\|_0$, so that the boundedness of $\lambda_{P,\ell}$ implies that w_ℓ is also bounded. Due to the reflexivity of $W \times Q$ there exists a subsequence $(w_{\tilde{\ell}}, \lambda_{P,\tilde{\ell}}) \subset (w_\ell, \lambda_{P,\ell})$ which weakly converges to $(w^*, \lambda^*) \in W \times \Lambda_P$. It is easy to see that $\lim_{\tilde{\ell}\to\infty} a(w_{\tilde{\ell}}, z_{\tilde{\ell}}) = a(w^*, z)$ and $\lim_{\tilde{\ell}\to\infty}(\mu_{\tilde{\ell}}, \chi(w_{\tilde{\ell}}))_0 = (\mu, \chi(w^*))$. Passing to the limit in Definition 2.2 yields

$$a(w^*, z) = \mathcal{F}(z) - (\lambda^*, \sigma_y q)_0, \tag{2.15}$$
$$(\mu, \chi(w^*))_0 \leq \liminf_{\tilde{\ell}\to\infty}(\lambda_{P,\tilde{\ell}}, \sigma_y w_{\tilde{\ell}})_0. \tag{2.16}$$

Since $z \mapsto a(z,z)$ is convex and continuous and, therefore, weakly lower semi-continuous, we obtain

$$\begin{aligned} a(w^*, w^*) + \liminf_{\tilde{\ell}\to\infty}(\lambda_{P,\tilde{\ell}}, \sigma_y(w_{\tilde{\ell}}))_0 &\leq \liminf_{\tilde{\ell}\to\infty}\Big(a(w_{\tilde{\ell}}, w_{\tilde{\ell}}) + (\lambda_{P,\tilde{\ell}}, \sigma_y(w_{\tilde{\ell}}))_0\Big) \\ &= \liminf_{\tilde{\ell}\to\infty}\mathcal{F}(w_{\tilde{\ell}}) = \mathcal{F}(w^*) \end{aligned}$$

from (2.3). Hence, using (2.15) with $z := w^*$ and (2.16), we find

$$(\mu, \sigma_y(w^*))_0 \leq \liminf_{\tilde{\ell}\to\infty}(\lambda_{P,\tilde{\ell}}, \sigma_y(w_{\tilde{\ell}}))_0 \leq \mathcal{F}(w^*) - a(w^*, w^*) = (\lambda^*, \sigma_y(w^*))_0. \tag{2.17}$$

Since (z, μ) is arbitrarily chosen, (2.15) and (2.17) imply that (w^*, λ^*) is a saddle point. Due to the uniqueness, we conclude $(w^*, \lambda^*) = (w, \lambda)$ and, additionally, that the entire sequence $(w_\ell, \lambda_{P,\ell})$ converges to (w, λ) weakly. To show that w_ℓ converges to w strongly, we conclude from (2.2)

$$\begin{aligned} a(w - w_\ell, w - w_\ell) &= a(w,w) - 2a(w, w_\ell) + \mathcal{F}(w_\ell) - (\lambda_{P,\ell}, \sigma_y(w_\ell))_0 \\ &\leq a(w,w) - 2a(w, w_\ell) + \mathcal{F}(w_\ell) - (\lambda, \sigma_y(w_\ell))_0 \to 0 \end{aligned}$$

as $\ell \to \infty$. □

2.2.2 Frictional Contact

In order to analyze the frictional contact problem for elastoplasticity with hardening, we use techniques known from the mixed formulation of frictional contact problems in linear elasticity. The following theorem is a modification of the Theorems 1.1.5.3 and 1.1.5.4 presented by Hlaváček et al. in [64].

Theorem 2.11. *Let the inf-sup condition* (2.12) *and the following assumptions be fulfilled.*

i. For all $z \in W$ there exists a sequence $z_\ell \in W_\ell$ strongly converging to z.

ii. For all $\mu \in \Lambda$ there exists a sequence $\mu_{\mathscr{L}} \in \Lambda_{\mathscr{L}}$ strongly converging to μ.

iii. For all sequences $\mu_{\mathscr{L}} \in \Lambda_{\mathscr{L}}$ weakly converging to μ, there holds $\mu \in \Lambda$.

iv. *There exists a bounded sequence* $\bar{z}_\ell \in K_{\ell,\mathscr{L}}$, *i.e.,*

$$\langle \mu_{C,\mathscr{L}}, \gamma(\bar{z}_\ell) - g\rangle \leq 0$$

holds for all $\mu_{C,\mathscr{L}} \in \Lambda_{C,\mathscr{L}}$.

Then, the sequence of discrete solutions w_ℓ *of the saddle point problem* (2.5) *converges strongly to the solution* w *of* (1.38) *and the sequence of discrete Lagrange multipliers* $\lambda_{\mathscr{L}}$ *converges weakly to the Lagrange multiplier* λ.

Proof. The ellipticity of a, the variational inequality (2.11), Condition *iv* and the boundedness of $\Lambda_{F,\mathcal{L}}$ yield

$$\begin{aligned}\nu_1\|w_\ell\|_W^2 &\leq a(w_\ell, w_\ell)\\ &\leq a(w_\ell, \bar{z}_\ell) + \Psi(\bar{z}_\ell) - \Psi(w_\ell) + \Psi_{F,\mathscr{L}}(\bar{z}_\ell) - \Psi_{F,\mathscr{L}}(w_\ell) - \mathcal{F}(\bar{z}_\ell - w_\ell)\\ &\leq (\nu_0\|\bar{z}_\ell\|_W + \sigma_y + \|\tilde{f}_F\gamma_t\| + \|\mathcal{F}\|)\|w_\ell\|_W + (\sigma_y + \|\tilde{f}_F\gamma_t\| + \|\mathcal{F}\|)\|\bar{z}_\ell\|_W.\end{aligned}$$

Hence, the sequence w_ℓ is bounded and therefore there exists a weakly convergent subsequence $w_{\hat{\ell}} \rightharpoonup \hat{w}$. From the inf-sup condition (2.12) and the discrete variational equality (2.8) we have

$$\begin{aligned}\alpha\|\lambda_{\mathscr{L}}\|_{(H^{-1/2}(\Gamma_C))^d} \leq \sup_{z_\ell\in W_\ell,\|z_\ell\|_W=1} b(\mu_{\mathscr{L}}, \gamma(z_\ell)) &= \sup_{(v_\ell,0)\in W,\|(v_\ell,0)\|_1=1} b(\mu_{\mathscr{L}}, \gamma(v_\ell, 0))\\ &\leq \nu_0\|w_\ell\|_W + \|\mathcal{F}\|.\end{aligned}$$

Thus, the sequence $\lambda_{\mathscr{L}}$ is also bound and we have a weakly convergent subsequence $\lambda_{\hat{\mathscr{L}}} \rightharpoonup \hat{\lambda}$. For arbitrary $z \in W$ and $\mu \in \Lambda$, we choose strongly convergent sequences $z_\ell \to z$ and $\mu_{\mathscr{L}} \to \mu$ and observe that

$$\begin{aligned}a(\hat{w}, \hat{w}) + \Psi(\hat{w}) - \mathcal{F}(\hat{w}) + \liminf b(\lambda_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}}))&\\ \leq \liminf a(w_{\hat{\ell}}, w_{\hat{\ell}}) + \Psi_{\hat{\ell}}(w_{\hat{\ell}}) - \mathcal{F}(w_{\hat{\ell}}) + b(\lambda_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}}))&\\ \leq \liminf a(w_{\hat{\ell}}, z_{\hat{\ell}}) + \Psi(z_{\hat{\ell}}) - \mathcal{F}(z_{\hat{\ell}}) + b(\lambda_{\hat{\mathscr{L}}}, \gamma(z_{\hat{\ell}}))&\\ = a(\hat{w}, z) + \Psi(z) - \mathcal{F}(z) + b(\hat{\lambda}, \gamma(z))&\end{aligned} \tag{2.18}$$

holds due to the weak lower semi-continuity of Ψ and $z \mapsto a(z, z)$ as well as the continuity of $\mathcal{F}$, a and Ψ. From the second inequality of the stationary condition (2.7), we have

$$b(\mu, \gamma(\hat{w}) - \tilde{g}) \leq \liminf b(\mu_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}}) - \tilde{g}) \leq \liminf b(\lambda_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}}) - \tilde{g})$$

Next, we choose $z = \hat{w}$ in (2.18) and conclude

$$b(\mu, \gamma(\hat{w}) - \tilde{g}) \leq \liminf b(\mu_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}}) - \tilde{g}) \leq \liminf b(\lambda_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}}) - \tilde{g}) \leq b(\hat{\lambda}, \gamma(\hat{w}) - \tilde{g}).$$

Hence, the choice $\mu = \hat{\lambda}$ in (1.39) implies

$$\begin{aligned} a(\hat{w}, \hat{w}) + \Psi(\hat{w}) - \mathcal{F}(\hat{w}) + b(\hat{\lambda}, \hat{w}) &\leq a(\hat{w}, \hat{w}) + \Psi(\hat{w}) - \mathcal{F}(\hat{w}) + \liminf b(\lambda_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}})) \\ &\leq a(\hat{w}, z) + \Psi(z) - \mathcal{F}(z) + b(\hat{\lambda}, \gamma(z))\rangle. \end{aligned}$$

This shows that $(\hat{w}, \hat{\lambda})$ fulfills (1.39) and in turn implies that $(\hat{w}, \hat{\lambda})$ is a saddle point of $\mathcal{L}_{CF}$. From the uniqueness of the saddle point, we conclude that the entire sequences w_ℓ and λ_ℓ weakly converge to $w = \hat{w}$ and $\lambda = \hat{\lambda}$, respectively.

Thus, it remains to show that w_ℓ even converges strongly. We note that

$$\begin{aligned} &a(w, w) + \Psi(w) + b(\lambda, \gamma(w)\rangle - \mathcal{F}(w) \\ &\quad \geq a(w, 2w - w) + \Psi(2w) - \Psi(w) + b(\lambda, \gamma(2w - w)) - \mathcal{F}(2w - w) \geq 0. \end{aligned}$$

Hence, with the stationary (2.6) and the complementary condition $b(\lambda, \gamma(w) - \tilde{g}) = 0$ we have

$$\begin{aligned} \nu_1 \|w - w_\ell\|_W^2 &\leq a(w - w_\ell, w - w_\ell) \\ &\leq a(w, w) - 2a(w, w_\ell) - \Psi(w_\ell) - b(\lambda_{\mathscr{L}}, \gamma(w_\ell)) + \mathcal{F}(w_\ell) \\ &\leq a(w, w) - 2a(w, w_\ell) - \Psi(w_\ell) - b(\mu_{\mathscr{L}}, \gamma(w_\ell) - \tilde{g}) - b(\lambda_{\mathscr{L}}, \tilde{g}\rangle + \mathcal{F}(w_\ell) \\ &\to -a(w, w) - \Psi(w) + b(\lambda, \gamma(w) - \tilde{g}) - b(\lambda, \gamma(w)) + \mathcal{F}(w) \leq 0 \end{aligned}$$

for $h \to 0$. $\square$

Next, we want to show that sequences of discrete saddle points $(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell})$ of the problem (2.13) also converge to the respective analytic solution. As in Remark 2.8, in addition to the case where $\Lambda_{P,\ell} \subset Q_\ell$, we consider the setting $\Lambda_{P,\ell} \not\subseteq Q_\ell$. We start with the convergence analysis for the first setting, i.e., we assume the set of discrete Lagrange multipliers $\Lambda_{P,\ell}$ to be a subset of Q_ℓ and bounded independently of the level ℓ. Then, the weak convergence of a sequence of discrete multipliers $\lambda_{P,\ell}$ follows directly from its uniqueness and the boundedness of $\Lambda_{P,\ell}$.

Theorem 2.12. *Let the assumptions of Theorem 2.11 hold. Additionally, Let hold the conditions ii and iii for $\Lambda_{P,\ell}$ and Λ_P, and $\Lambda_{P,\ell} \subset Q_\ell$ be bounded independently of ℓ. Then, the sequences of discrete Lagrange multipliers $\lambda_{P,\ell}$ and $\lambda_{\mathscr{L}}$ weakly converge to the Lagrange multipliers λ_P and λ, and w_ℓ converges strongly to w.*

Proof. As in the proof of the previous theorems, we conclude that the sequences w_ℓ, $\lambda_{\mathscr{L}}$ and $\lambda_{P,\ell}$ are bounded. Let $\hat{w}$, $\hat{\lambda}$ and $\hat{\lambda}_P$ be the weak limits of the weakly convergent subsequences $w_{\hat{\ell}}$, $\lambda_{\hat{\mathscr{L}}}$ and $\lambda_{P,\hat{\ell}}$, respectively. Now, for $z \in W$, $\mu \in \Lambda$ and $\mu_P \in \Lambda$ arbitrary, we choose strongly convergent sequences $z_\ell \to z$, $\mu_{\mathscr{L}} \to \mu$ and $\mu_{P,\ell} \to \mu_P$ which exist due to the assumption *i*. From the stationary condition (2.13) and the continuity of the linear and bilinear forms, we have

$$\begin{aligned} a(\hat{w}, z) = \lim a(w_{\hat{\ell}}, z_{\hat{\ell}}) &= \lim \mathcal{F}(z_{\hat{\ell}}) - b(\lambda_{\hat{\mathscr{L}}}, \gamma(z_{\hat{\ell}})) - (\lambda_{P,\hat{\ell}}, \sigma_y q_{\hat{\ell}})_0 \\ &= \mathcal{F}(z) - b(\hat{\lambda}, \gamma(z)) - (\hat{\lambda}_P, \sigma_y q)_0 \end{aligned}$$

and

$$b(\mu, \gamma(\hat{w}) - \tilde{g}) + (\mu_P, \sigma_y \hat{p})_0 \leq \liminf \big(b(\lambda_{\hat{\mathscr{L}}}, \gamma(w_{\hat{\ell}}) - \tilde{g}) + (\lambda_{P,\hat{\ell}}, \sigma_y p_{\hat{\ell}})_0 \big).$$

For $z = \hat{w}$ we conclude

$$\begin{aligned} b(\mu, \gamma(\hat{w}) - \tilde{g}) + (\mu_P, \sigma_y \hat{p})_0 &\leq b(\hat{\lambda}, -\tilde{g}) + \liminf \big(\mathcal{F}(w_{\hat{\ell}}) - a(w_{\hat{\ell}}, w_{\hat{\ell}}) \big) \\ &= b(\hat{\lambda}, \gamma(\hat{w}) - \tilde{g}) + (\hat{\lambda}_P, \sigma_y \hat{p})_0. \end{aligned}$$

Altogether, this shows that $(\hat{w}, \hat{\lambda}, \hat{\lambda}_P) \in W \times \Lambda \times \Lambda_P$ is a saddle point of $\mathcal{L}$. Since $\mathcal{L}$ has exactly one saddle point even the entire sequences weakly converge to $(\hat{w}, \hat{\lambda}, \hat{\lambda}_P)$ and it only remains to show that the sequence w_ℓ converges strongly to the first component $\hat{w}$. To this end, we observe

$$\begin{aligned} \nu_0 \|w - w_\ell\|_W &\leq a(w - w_\ell, w - w_\ell) \\ &= a(w, w) - 2a(w, w_\ell) + \mathcal{F}(w_\ell) - b(\lambda_{\mathscr{L}}, \gamma(w_\ell)) - (\lambda_{P,\ell}, \sigma_y p_\ell)_0. \end{aligned}$$

Hence,

$$\|w - w_\ell\|_W \to 0$$

as $\ell \to \infty$. □

If $\Lambda_{P,\ell} \not\subseteq Q_\ell$, we assume the inf-sup condition (2.14) to hold for all discrete multipliers. Hence, the weak convergence of $\lambda_{P,\ell}$ follows in the same way as for the discrete Lagrange multipliers $\lambda_{C,\mathscr{L}}$ and $\lambda_{F,\mathscr{L}}$. As it is easy to see the arguments of the proof of Theorem 2.11 still hold when the variational inequality (2.6) is replaced by the variational equality of the stationary condition (2.13). Actually, the argumentation becomes even simpler.

Theorem 2.13. *Let $\Lambda_{P,\ell} \not\subseteq Q_\ell$ and the discrete inf-sup condition* (2.14) *be fulfilled and let hold the conditions i- iv of Theorem 2.11. Additionally, let hold the conditions ii and iii for $\Lambda_{P,\ell}$ and Λ_P. Then, the sequence of the discrete Lagrange multipliers $\lambda_{P,\ell}$ weakly converges to λ_P and w_ℓ converges strongly to w.*

Proof. The boundedness of the sequences is a direct consequence of the discrete inf-sup condition (2.14). Now, we can follow the arguments of the proof of the previous theorem. □

2.3 Finite Elements for the Mixed Formulations

In this section, we present some particular discretization spaces based on finite elements. Moreover, we use the results of the previous sections to show unique existence of discrete saddle points based on finite elements. Moreover, we proof their convergence to the analytic solutions. The finite elements are based on meshes consisting of triangles or quadrilaterals in two dimensions and of hexahedrons in three dimensions. The results for meshes of quadrilaterals also hold if the same mesh additionally contains triangular cells. The basis functions are chosen to be polynomials of arbitrary degree. This way, h- as well as p-refinement is possible. To conclude the uniqueness of the Lagrange multipliers we will use that the inf-sup condition (2.12) holds if the quotient hm^2/Hl is sufficiently small. Here, l and m denote the polynomial degree, and h and H the mesh size of V_ℓ and $M_{\mathscr{L}}$, respectively. However, in practice it remains to determine what sufficiently small means.

In the case that the inf-sup condition is not fulfilled the existence of the multiplier holds nevertheless and solution algorithms can be able to find random multipliers. This can lead to Lagrange multipliers without a physical meaning as can be seen in e.g. [87]. Whereas the uniqueness of the first component w_ℓ is still guaranteed as it is the minimizer of an energy functional like in the infinite dimensional setting.

Let $\mathcal{T}_\ell$ be a shape regular mesh of the considered domain Ω consisting of triangles and/or quadrilaterals in two dimensions and of hexahedrons in three dimensions. From here on, we assume the domain Ω to have a polygonal boundary and the mesh $\mathcal{T}_\ell$ to exactly cover the whole domain. Moreover, we restrict the quadrilaterals and hexahedrons to be parallelograms and parallelepipeds, respectively. A mesh is said to be regular if the intersection of two non identical elements is either empty, a node, or a complete edge or face. As usual, we call a sequence of meshes non degenerated if for each element the radius of the biggest inner ball can be bounded by a constant times the diameter and the constant does neither depend on the element nor the level. Moreover, we use constraints coefficients, see [88, 91], in order to include hanging nodes into the mesh. By this means, we will be able to locally refine meshes of quadrilaterals and construct adaptive finite element schemes based on the a posteriori estimates of Chapter 4. If a mesh consists of triangles, we demand that it is regular. Note that, a mesh with hanging nodes is not regular.

The different parts of the boundary are assumed to be resolved exactly by the mesh, i.e., for all cells $K \in \mathcal{T}_\ell$ we assume $K \cap \Gamma_\Upsilon$ to be an entire edge, a vertex or empty for a fixed $\Upsilon = N, C, D$. We denote the diameter of a cell $K \in \mathcal{T}_\ell$ by h_K. Moreover, we define the maximal diameter $h := \max_{K\in\mathcal{T}_\ell} h_K$ which we will also refer to as mesh size in the following. Further, let $\mathcal{T}_{\mathscr{L}}$ be a shape regular mesh of Γ_C with mesh size H and H_K the diameter of an edge or face $K \in \mathcal{T}_{\mathscr{L}}$. The volume of a cell, face or edge K is denoted by $|K|$. The set of nodes of $\mathcal{T}_\ell$ and $\mathcal{T}_{\mathscr{L}}$ is denoted by $\mathcal{N}_\ell$ and $\mathcal{N}_{\mathscr{L}}$, respectively, where we do not include hanging nodes. The edges contained in a subset $\omega \subset \mathcal{T}_\iota$ is denoted by $\mathcal{E}_\iota(\omega)$ for $\iota \in \{\ell, \mathscr{L}\}$. For the set of all edges or faces of a mesh $\mathcal{T}_\iota$ we write $\mathcal{E}_\iota$ and for the interior edges or faces $\mathcal{E}_\ell^\circ$. The edges or faces of $\mathcal{T}_\ell$ on the different parts of the boundary Γ_D, Γ_N and Γ_C are denoted by $\mathcal{E}_\ell^D$, $\mathcal{E}_\ell^N$ and $\mathcal{E}_\ell^C$, respectively. Let $\mathcal{P}^k(T)$ denote the space of polynomials of partial degree at most k if T is a quadrilateral or hexahedron and of degree at most k if T is a triangle or edge.

In a first step, we introduce the standard spaces of piecewise polynomials

$$\mathcal{P}_h^k(\Omega) := \{v : \Omega \to \mathbb{R} \mid v_{|T} \circ \Phi_T \in \mathcal{P}^k(T_{\text{ref}}), T \in \mathcal{T}_\ell(\Omega)\}$$

and
$$\mathcal{P}_H^m(\Gamma_C) := \{v : \Gamma_C \to \mathbb{R} \mid v_{|T} \circ \Phi_T \in \mathcal{P}^m(E_{\text{ref}}), T \in \mathcal{T}_{\mathscr{L}}(\Gamma_C)\}$$
where Φ_T is the transformation from reference cell to the actual cell.

Now, we introduce the discrete spaces as

$$\begin{aligned} V_h^l &:= \{v \in V \mid v_i \in \mathcal{P}_h^l(\Omega)\}, \\ Q_h^k &:= \{q \in Q \mid q_{ij} \in \mathcal{P}_h^k(\Omega)\}, \\ W_h^{l,k} &:= V_h^l \times Q_h^k, \\ M_H^m &:= \{\mu \in L^2(\Gamma_C, \mathbb{R}) \mid \mu \in \mathcal{P}_H^m(\Gamma_C)\}. \end{aligned}$$

Thus, the spaces M_H^m and Q_h^k consist of piecewise polynomials allowed to be discontinuous over nodes, edges or faces, whereas the functions in V_h^l are continuous on the entire domain Ω. The direct approach to a discrete variational inequality usually involves triangles as well as piecewise affine and constant functions, see for example [57, 26]. This setting is obviously included in the discretization spaces presented above. However, the choice of polynomial degrees bigger than one for elastoplasticity with linear hardening has not yet been investigated to a great extend. Some results concerned with the p-version of the finite element method for variational inequalities are found in [74]. The analysis of the mixed formulation for higher polynomials degrees seems to be new since the discretization scheme is no longer guaranteed to be equivalent to the discrete variational inequality.

For meshes of triangles and $w_\ell \in W_h^{l,l-1}$, $l = 1, 2 \ldots$, we observe that

$$\operatorname{dev}(\sigma(w_\ell)) \in Q_h^{l-1}.$$

This property will play an important role in the proof of the convergence for the adaptive finite element method in Chapter 5. In order to guarantee this property for meshes of quadrilaterals, we set

$$\hat{Q}_h^l := \{q \in Q \mid \exists v \in V_h^l \; \forall T \in \mathcal{T} \; q_{|T} = \operatorname{dev}(\sigma(v_{|T}))\}.$$

Obviously, $\operatorname{dev}(\sigma(w_\ell)) \in Q_\ell$ holds for all $w_\ell \in \hat{W}_h^{l,l} := V_h^l \times \hat{Q}_h^l$.

It remains to define the sets of discrete Lagrange multipliers. If the degree m of the space M_H^m is zero or one we set

$$\Lambda_{C,H}^m := \{\mu_C \in M_H^m \mid \mu_C(\xi) \geq 0 \text{ for all } \xi \in \mathcal{N}_H\}, \tag{2.19}$$
$$\Lambda_{F,H}^m := \{\mu_F \in (M_H^m)^{d-1} \mid \mu_F(\xi)^\top \mu_F(\xi) \leq \zeta(f_F, \tilde{f}_F) \text{ for all } \xi \in \mathcal{N}_H\}. \tag{2.20}$$

This definition implies $\Lambda_{C,H} \subset \Lambda_C$ and $\Lambda_{F,H} \subset \Lambda_F$ and thereby defines a conforming ansatz. For polynomial degrees $m > 1$ a definition of the sets of Lagrange Multipliers as subsets of the sets Λ_C and Λ_F is not obvious since restrictions for single points do no longer imply the desired restrictions for almost all points. We define $\mathcal{G}^{d-1}(E)$ as the set of the $(m+1)^{d-1}$ transformed points of the Gaussian quadrature on the edge E. In order to define sets of discrete Lagrange multipliers for higher order finite elements, we demand the conditions $\mu_n(\xi) \geq 0$ and $\mu_t(\xi)^\top \mu_t(\xi) \leq \zeta(f_F, \tilde{f}_F)$ to be fulfilled for $\xi \in \mathcal{G}^{d-1}(E)$ and for all $E \in \mathcal{E}_H(\Gamma_C)$. We abbreviate $\mathcal{G}^{d-1} = \bigcup_{E \in \mathcal{E}(\Gamma_C)} \mathcal{G}^{d-1}(E)$ and define

$$\Lambda^m_{C,H} := \{\mu_C \in M^m_H \mid \mu_C(\xi) \geq 0 \text{ for all } \xi \in \mathcal{G}^{d-1}\}, \tag{2.21}$$

$$\Lambda^m_{F,H} := \{\mu_F \in (M^m_H)^{d-1} \mid \mu_F(\xi)^\top \mu_F(\xi) \leq \zeta(f_F, \tilde{f}_F) \text{ for all } \xi \in \mathcal{G}^{d-1}\} \tag{2.22}$$

for degrees $m > 1$. We assume that the support of the frictional resistance function is resolved by the mesh and that the additionally induced quadrature error is negligible.

The set of discrete multipliers for the nonlinearity arising from elastoplasticity is defined in a similar fashion as the set $\Lambda^m_{F,H}$. So in order to discretize Λ_P, we define

$$\Lambda^k_{P,h} := \{\mu_P \in Q^k_h \mid \mu_P(\xi) : \mu_P(\xi) \leq 1 \text{ for all } \xi \in \mathcal{N}_h\} \tag{2.23}$$

for degrees $k \leq 1$ and

$$\Lambda^k_{P,h} := \{\mu_P \in Q^k_h \mid \mu_P(\xi) : \mu_P(\xi) \leq 1 \text{ for all } \xi \in \mathcal{G}^d\} \tag{2.24}$$

for degrees $k > 1$. Here, $\mathcal{G}^d(K)$ is the set of transformed points for the Gaussian quadrature on K and $\mathcal{G}^d := \bigcup_{K \in \mathcal{T}_h(\Omega)} \mathcal{G}^d(K)$. With this definitions the same nonconformity as for the contact multipliers occurs, i.e., the inclusion $\Lambda^k_{P,h} \subset \Lambda_P$ does not hold for $k > 1$. Whereas, we have $\Lambda^k_{P,h} \subset Q^k_h$ for all polynomial degrees k. For ease of notation we also write $q_\ell = q^k_h$, $Q_\ell := Q^k_h$, $\mu_{\mathscr{L}} = \mu^m_H$ $\Lambda_{\mathscr{L}} := \Lambda^m_H := \Lambda^m_{C,H} \times \Lambda^m_{F,H}$, $\Lambda_{P,\ell} := \Lambda^k_{P,h}$, etc.

This choice of the discretization for Λ will be crucial to show that the convergence results of Section 2.2 hold for the discretization schemes based on the finite element spaces. For now, we only focus on the inf-sup condition, which will be shown to hold on the whole spaces M^m_H and $(M^m_H)^{d-1}$ rather than the sets of multipliers.

In order to establish the inf-sup condition for (1.38) we consider the auxiliary problem of finding $w_\mu := (u_\mu, 0)$ such that

$$a(w_\mu, z) = b(\mu, \gamma(v)) \tag{2.25}$$

holds for all $z := (v, 0)$, where $\mu = (\mu_C, \mu_F) \in (H^{-1/2}(\Gamma_C))^d$. This problem is said to be regular, if for all $\mu \in H^{-1/2+\theta}(\Gamma_C)$ holds $u_\mu \in H^1_D(\Omega) \cap H^{1+\theta}(\Omega)$ and $\|u_\mu\|_{1+\theta} \leq C\|\mu\|_{-1/2+\theta,\Gamma_C}$ with $0 < \theta \leq 1/2$. Since the inf-sup-condition (2.12) only depends on the choice of V^l_h and M^m_H we are able to use a known result from Schröder et al. [98], which holds if the frictional resistance is included in the variational formulation rather than the of Lagrange multipliers.

Theorem 2.14 ([98, Theorem 9])**.** *Let the problem* (2.25) *be regular,* $\tilde{f}_F = f_F$*, and*

$$(hH^{-1} \max\{1, m\}^2 l^{-1})^\theta$$

be sufficiently small then there exists an $\alpha > 0$ *independent of* h*,* H*,* m *and* l *such that the inf-sup condition* (2.12) *holds for all* $(\mu_{C,H}, \mu_{F,H}) \in M_H^m \times (M_H^m)^{d-1}$.

As mentioned earlier, in practice it is not clear what sufficiently small means in actual numbers. Thus, the assumptions in Theorem 2.14 are rather of theoretical nature and therefore, it is necessary to heuristically determine the right ratio of degrees and mesh size by numerical experiments. Also the regularity assumption on problem (2.25) is only known to be fulfilled in some special cases. Nevertheless, the theorem shows that the choice of the discretization spaces is reasonable since an important assumption for uniqueness and convergence of discrete saddle points can be fulfilled. Hence, to proof the convergence of finite element schemes based on the chosen spaces, it remains for us to verify that the conditions *i* - *iv* of Theorem 2.11 hold.

The condition *i* holds for the space V_h^l due to the interpolation results in [86]. In [98], the conditions *ii*, *ii* and *iv* are verified for the set of Lagrange multipliers $\Lambda_{C,H}^m$. We will show that the choice of $\Lambda_{F,H}^m$ for the Lagrange multipliers associated with friction also fulfills the conditions. The next Lemma states this for $d = 2, 3$ and arbitrary polynomial degrees m whereas the proof in [93] only holds for $d = 2$. In [64], the convergence is shown for $\Lambda_{\mathscr{L}} \subset \Lambda$ only. We denote by C_D^∞ the arbitrarily smooth functions on Ω which are zero on Γ_D

Lemma 2.15. *Let* $\tilde{f}_F$ *be such that* $\tilde{f}_F \gamma_t(z) \in C(\Gamma_C)$ *for all* $z \in C_D^\infty(\Omega)$*. For* $\Lambda_{F,H}^m$ *as defined in* (2.20)*,* (2.22) *the conditions* *ii*- *iii* *of Theorem 2.11 hold.*

Proof. For $\theta > 0$, we have that $H^{1+\theta}(\Gamma_C; \mathbb{R}^{d-1}) \cap \Lambda_F$ is dense in Λ_F. Hence, for all $\mu_F \in \Lambda_F$ and $\epsilon > 0$ there exists a $\mu_\epsilon \in H^{1+\theta}(\Gamma_C; \mathbb{R}^{d-1}) \cap \Lambda_F$ such that $\|\mu - \mu_\epsilon\| < \epsilon/2$. We choose the level $\mathscr{L}$ (i.e. polynomial degree m and mesh size H) such that $\|\mu_\epsilon - I_{\mathscr{L}}(\mu_\epsilon)\| < \epsilon/2$ and set $\mu_{F,\mathscr{L}} := I_{\mathscr{L}}(\mu_\epsilon)$ with $I_{\mathscr{L}}$ the usual interpolant using the set of transformed Gauss points $\mathcal{G}^{d-1}$. Altogether, we have $\|\mu - \mu_{F,\mathscr{L}}\| \leq \|\mu - \mu_\epsilon\| + \|\mu_\epsilon - \mu_{F,\mathscr{L}}\| < \epsilon$ which shows condition *ii*.

Now, let $\mu_{F,\mathscr{L}} \rightharpoonup \mu_F$. For $\tilde{f}_F \gamma_t(z) \in C(\Gamma_C)$ we have $I_{\mathscr{L}}(\tilde{f}_F \gamma_t(z)) \to \gamma_t(z)$ strongly in $H^{1/2}(\Gamma_C)$. For all $z \in C_D^\infty(\Omega; \mathbb{R}^d) \times Q$, we have

$$
\begin{aligned}
(\mu_F, \tilde{f}_F \gamma_t(z))_0 &= \lim (\mu_{F,\mathscr{L}}, \tilde{f}_F \gamma_t(z))_0 \\
&= \lim (\mu_{F,\mathscr{L}}, \tilde{f}_F \gamma_t(z) - I_{\mathscr{L}}(\tilde{f}_F \gamma_t(z)))_0 + \lim (\mu_{F,\ell}, I_{\mathscr{L}}(\tilde{f}_F \gamma_t(z)))_0 \\
&= \lim \sum_{\xi \in \mathcal{G}^{d-1}} \alpha_\xi \mu_{F,\mathscr{L}}(\xi)^\top \tilde{f}_F(\xi) \gamma_t(z)(\xi) \\
&\leq \lim \sum_{\xi \in \mathcal{G}^{d-1}} \alpha_\xi (\gamma_t(z)(\xi)^\top \tilde{f}_F(\xi) \gamma_t(z)(\xi))^{1/2} \\
&= \int_{\Gamma_C} (\gamma_t(z)^\top \tilde{f}_F \gamma_t(z))^{1/2}.
\end{aligned}
$$

And the density of $C_D^\infty(\Omega; \mathbb{R}^d) \times Q$ in W gives $(\mu_F : \mu_F) \leq \zeta(f_F, \tilde{f}_F)$ a.e on Γ_C. □

In addition, the approximation property *i* for $q \in Q$ by functions $q_\ell \in Q_\ell$ is standard and can be found in e.g. [86]. If we use three Lagrange multipliers, as in the stationary condition (2.13), we still have to verify conditions *ii* and *iii* for the set $\Lambda_{P,\ell}$. This is done in a similar fashion as for the set $\Lambda^m_{F,H}$.

Lemma 2.16. *For Q_ℓ and $\Lambda_{P,\ell}$ as defined in* (2.23)*,* (2.24) *the conditions ii- iii of Theorem 2.11 hold.*

Proof. For $\theta > 0$, we have that $H^{1+\theta}(\Omega;\mathbb{R}^{d\times d}) \cap \Lambda_P$ is dense in Λ_P. Hence, for all $\mu_P \in \Lambda_P$ and $\epsilon > 0$ there exists a $\mu_\epsilon \in H^{1+\theta}(\Omega;\mathbb{R}^{d\times d}) \cap \Lambda_P$ such that $\|\mu - \mu_\epsilon\| < \epsilon/2$. We choose the level ℓ (i.e. polynomial degree k and mesh size h) such that $\|\mu_\epsilon - I_\ell(\mu_\epsilon)\| < \epsilon/2$ and set $\mu_{P,\ell} := I_\ell(\mu_\epsilon)$ with I_ℓ the usual interpolant using the set of transformed Gauss points $\mathcal{G}^d$. Altogether, we have $\|\mu - \mu_{P,\ell}\| \le \|\mu - \mu_\epsilon\| + \|\mu_\epsilon - \mu_{P,\ell}\| < \epsilon$ which shows condition *ii*.

Now, let $\mu_{P,\ell} \rightharpoonup \mu_P$. For all $q \in H^{1+\theta}(\Omega;\mathbb{R}^{d\times d}) \cap Q$, we have

$$
\begin{aligned}
(\mu_P, \sigma_y q)_0 = \lim(\mu_{P,\ell}, \sigma_y q)_0 &= \lim(\mu_{P,\ell}, \sigma_y(q - I_\ell(q)))_0 + \lim(\mu_{P,\ell}, \sigma_y I_\ell(q))_0 && (2.26)\\
&= \lim \sum_{\xi\in\mathcal{G}^d} \alpha_\xi \sigma_y \mu_{P,\ell}(\xi) : q(\xi) && (2.27)\\
&\le \lim \sum_{\xi\in\mathcal{G}^d} \alpha_\xi (q(\xi) : \sigma_y q(\xi))^{1/2} = \int_\Omega (q : \sigma_y q)^{1/2}. && (2.28)
\end{aligned}
$$

And the density of $H^{1+\theta}(\Omega;\mathbb{R}^{d\times d})$ in Q gives $(\mu_P : \sigma_y \mu_P) \le 1$ a.e in Ω. □

Altogether, the choice $W_\ell := W^{l,k}_h$, $\Lambda_{\mathscr{L}} := \Lambda^m_{C,H} \times \Lambda^m_{F,H}$ and $\Lambda_{P,\ell} := \Lambda^k_{P,h}$ allows for the construction of well defined and convergent finite element methods. Moreover, the restriction on the Lagrange multipliers is easy to handle within numerical solution schemes. For example, if we choose Lagrange basis functions based on the given points then the conditions are applied directly to the coefficient vector.

3 A priori estimates

For variational equalities, the Galerkin orthogonality usually provides the quasi best approximation of the numerical approximation. From this property, classic results on polynomial interpolation yield results on the rates of convergence. In the case of variational inequalities, the Galerkin orthogonality no longer holds and hence we have to investigate the speed of convergence via different approaches. Within this chapter we will combine well known a priori estimates for elastoplasticity with hardening and error estimates for the dual mixed formulation of contact problems. This combination results in a priori estimates for the contact problem in elastoplasticity with linear kinematic hardening.

In the following, $A \lesssim B$ abbreviates $A \leq CB$ with a positive constant C which is independent of W_ℓ and $\Lambda_{\mathscr{L}}$ whereas it can depend on the material parameters and especially the hardening tensor $\mathbb{H}$. Furthermore, $A \approx B$ represents $A \lesssim B \lesssim A$. As before, we denote the solution of the saddle point problem (1.39) by (w, λ) and the solution of the discrete problem (2.6),(2.7) by $(w_\ell, \lambda_{\mathscr{L}})$.

In Section 3.1, we derive bounds on the error which hold independent of the actual discretization spaces. This bounds establish some sort of best approximation property. As another consequence the estimates yield the strong convergence of the Lagrange multipliers. Section 3.2 contains the derivation of rates of convergence for the finite element spaces introduced in the previous Chapter. In order to obtain the conclusions on the convergence speed, we combine the results of the first section with well known interpolation results. Both sections each start with the investigation of the elastoplastic problem without contact conditions of Definition 1.1 then followed by results for the frictional contact problem of Definition 1.18. The convergence rates give estimates on the error with respect to the mesh sizes and the polynomial degrees. Thus, in principle, they are valid for both the h- as well as the p-method.

3.1 A Priori Bounds

The main results of this section are a priori estimates for the discretization errors of the saddle point problems introduced above. The estimates are not based on the finite element spaces introduced in the previous chapter but arbitrary discrete spaces. Although in some cases, we will use additional properties of the discretization spaces.

3.1.1 Elastoplasticity

Once again, we begin with the analysis of the discretization of the model of elastoplasticity with linear kinematic hardening without any restrictions due to contact or friction. We focus on the stationary condition (1.17) for the saddle point (w, λ_P) and its discrete approximation $(w_\ell, \lambda_{P,\ell}) \in W_\ell \times \Lambda_{P,\ell}$. Recall that for polynomial degrees greater than one we do not assume $\Lambda_{P,\ell} \not\subseteq \Lambda_P$, which is a crucial assumption in [97]. Nevertheless, we will redisplay a result of [97], since we can use it in the case that we use polynomial degrees smaller one.

Theorem 3.1 ([97, Theorem 5.2]). *Let $\Lambda_{P,\ell} \subset \Lambda_P$ and $\Lambda_{P,\ell} \subset Q_\ell$. For the saddle point (w, λ_P) of $\mathcal{L}_P$ and its discrete approximation $(w_\ell, \lambda_{P,\ell})$, there holds*

$$\|w - w_\ell\|_W + \|\lambda_P - \lambda_{P,\ell}\|_0 \lesssim \|w - z_\ell\|_W + \|\lambda_P - \mu_{P,\ell}\|_0 + (\lambda_P - \mu_{P,\ell}, \chi(w))_0^{1/2} \quad (3.1)$$

for all $z_\ell \in W_\ell$ and all $\mu_{P,\ell} \in \Lambda_{P,\ell}$.

If we follow the ideas in [93] we can derive some sort of a priori estimate even if we drop the assumption $\Lambda_{P,\ell} \subset \Lambda_P$. In the next section, we will address the problems which arise when we want to derive more specific bounds. However, the estimate together with some further assumptions will imply the strong convergence of the sequence of discrete Lagrange multipliers. We start with a result concerning the connection between the discretization error for the Lagrange multiplier and the one of the primal variables.

Lemma 3.2. *Let $\Lambda_{P,\ell} \subset Q_\ell$ then there holds*

$$\|\lambda_P - \lambda_{P,\ell}\|_0 \lesssim \|w - w_\ell\|_W + \|\lambda_P - \mu_{P,\ell}\|_0$$

for all $\mu_{P,\ell} \in \Lambda_{P,\ell}$

Proof. We obtain

$$\begin{aligned}
\sigma_y \|\mu_{P,\ell} - \lambda_{P,\ell}\|_0 &= \sup_{q_\ell \in Q_\ell,\ \|q_\ell\|_0 = 1} (\mu_{P,\ell} - \lambda_{P,\ell}, \sigma_y q_\ell)_0 \\
&= \sup_{q_\ell \in Q_\ell,\ \|q_\ell\|_0 = 1} \big((\mu_{P,\ell}, \sigma_y q_\ell)_0 + a(w_\ell, (0, q_\ell)) - \mathcal{F}((0, q_\ell))\big) \\
&= \sup_{q_\ell \in Q_\ell,\ \|q_\ell\|_0 = 1} \big((\mu_{P,\ell} - \lambda_P, \sigma_y q_\ell)_0 + a(w_\ell - w, (0, q_\ell))\big) \\
&\le \sigma_y \|\lambda_P - \mu_{P,\ell}\|_0 + \nu_0 \|w - w_\ell\|_W.
\end{aligned}$$

Thus, we have

$$\|\lambda - \lambda_\ell\|_0 \le \|\lambda - \mu_\ell\|_0 + \|\mu_\ell - \lambda_\ell\|_0 \le 2\|\lambda - \mu_\ell\|_0 + \nu_0 \sigma_y^{-1} \|w - w_\ell\|_W.$$

$\square$

Now, we are able proof an estimate for the combined discretization error of the primal variable and the Lagrange multiplier.

Theorem 3.3. *Let $\Lambda_{P,\ell} \subset Q_\ell$. There exist constants $c_1, c_2 > 0$ such that*

$$\begin{aligned}\|w - w_\ell\|_W^2 + \|\lambda_P - \lambda_{P,\ell}\|_0^2 \leq & c_1\Big(\|w - z_\ell\|_W^2 + \|\lambda_P - \mu_{P,\ell}\|_0^2\Big) \\ & + c_2\Big((\lambda_P - \mu_{P,\ell}, \sigma_y p)_0 + (\lambda_{P,\ell} - \mu_P, \sigma_y p)_0\Big)\end{aligned}$$

holds for all $z_\ell \in W_\ell$, $\lambda_{\mathscr{L}} \in \Lambda$ and $\lambda_{P,\ell} \in \Lambda_P$.

Proof. We use the variational equation in the stationary condition (1.17) and its discrete counterpart (2.3) to conclude that

$$\begin{aligned}a(w - w_\ell, w - w_\ell) &= a(w - w_\ell, w - z_\ell) + a(w - w_\ell, z_\ell - w_\ell) \\ &= a(w - w_\ell, w - z_\ell) + (\lambda_{P,\ell} - \lambda_P, q_\ell - p)_0 + (\lambda_{P,\ell} - \lambda_P, p - p_\ell))_0.\end{aligned}$$

Moreover, from the inequality in the stationary condition (1.17), we have

$$(\lambda_{P,\ell} - \lambda_P, p - p_\ell)_0 \leq (\lambda_{P,\ell} - \mu_P, p)_0 + (\lambda_P - \mu_{P,\ell}, p)_0 + (\lambda_P - \mu_{P,\ell}, p - p_\ell)_0.$$

The ellipticity and continuity of the bilinear form a, and the Cauchy-Schwarz inequality lead to

$$\begin{aligned}\nu_1\|w - w_\ell\|_W^2 \leq \nu_0\|w - w_\ell\|_W\|w - z_\ell\|_W + \|\lambda_P - \mu_{P,\ell}\|_0\|w - w_\ell\|_W & \\ + (\lambda_{P,\ell} - \mu_P, \sigma_y p)_0 + (\lambda_P - \mu_{P,\ell}, \sigma_y p)_0.&\end{aligned}$$

Finally, Lemma 3.2 and Young's inequality yield the assertion. □

In order to deduce the strong convergence of the discrete Lagrange multiplier, we use density assumptions like the conditions in Chapter 2.

Corollary 3.4. *Let hold the assumptions of Theorem 2.10. Then*

$$\lambda_{P,\ell} \to \lambda_P$$

strongly in Q_ℓ

Proof. From conditions *i* and *ii*, we can choose z_ℓ and $\mu_{P,\ell}$ such that the first three summands of the right hand side in Theorem 3.3 converge to zero as $\ell \to \infty$. Next, we choose $\mu_P = \lambda_P$. Thus, also the fourth summand tends to zero due to the weak convergence of $\lambda_{P,\ell}$. □

3.1.2 Frictional Contact

After the focus on the purely elastoplastic model, we return to the analysis of the Problem of Section 1.5. We will look at both saddle point formulations the one based on two Lagrange multipliers for contact and friction, and the one which includes a third multiplier for the plastic dissipation functional.

The first result presented in this subsection bounds the error in the Lagrange multipliers for contact and friction with the error in the primal variables for displacement and plastic strain as well as the distance of $\Lambda_{\mathscr{L}}$ to the analytic multiplier λ. This error enables us to include the discretization error of the Lagrange multipliers into the main results.

Lemma 3.5. *Let the inf-sup condition* (2.12) *be fulfilled. It holds*

$$\|\lambda - \lambda_{\mathscr{L}}\|_{-1/2} \lesssim \|w - w_\ell\|_W + \|\lambda - \mu_{\mathscr{L}}\|_{-1/2}$$

for all $\mu_{\mathscr{L}} \in \Lambda_{\mathscr{L}}$.

Proof. Due to the inf-sup condition (2.12) and the discrete variational equality (2.8) we have

$$\begin{aligned}
\alpha\|\mu_{\mathscr{L}} - \lambda_{\mathscr{L}}\|_{-1/2} &\leq \sup_{\substack{z_\ell \in W_\ell,\\ \|z_\ell\|_W = 1}} b(\mu_{\mathscr{L}} - \lambda_{\mathscr{L}}, \gamma(z_\ell)) \\
&= \sup_{\substack{v_\ell \in V_\ell,\\ \|v_\ell\|_1 \leq 1}} b(\mu_{\mathscr{L}} - \lambda_{\mathscr{L}}, \gamma((v_\ell, 0))) \\
&= \sup_{\substack{v_\ell \in V_\ell,\\ \|v_\ell\|_1 \leq 1}} b(\mu_{\mathscr{L}}, \gamma((v_\ell, 0))) + a(w_\ell, (v_\ell, 0)) - \mathcal{F}((v_\ell, 0)) \\
&= \sup_{\substack{v_\ell \in V_\ell,\\ \|v_\ell\|_1 \leq 1}} b(\mu_{\mathscr{L}} - \lambda, \gamma((v_\ell, 0))) + a(w_\ell - w, (v_\ell, 0)) \\
&\leq \|(\gamma_n, \tilde{f}_F \gamma_t)\|_{L(W,(H^{1/2}(\Gamma_C))^d)} \|\mu_{\mathscr{L}} - \lambda\|_{-1/2} + \nu_0 \|w - w_\ell\|_W.
\end{aligned}$$

This and the application of the triangle inequality conclude the proof. □

Since this result only depends on the inf-sup condition (2.12) and the equation (2.8), it holds for both mixed formulations considered in this section. Although we included both alternatives for definition of the Lagrange multiplier for friction, we note that Theorem 2.14 only guarantees the inf-sup condition for the case where the norm of the multiplier is bounded by one. As before, we start to analyze the approximation scheme for the saddle point problem (1.38) where only the contact and friction conditions are treated by Lagrange multipliers.

Theorem 3.6. *Let the inf-sup condition* (2.12) *be fulfilled. There exist constants $c_1, c_2 > 0$ such that*

$$\begin{aligned}
\|w - w_\ell\|_W^2 + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2}^2 \leq c_1\Big(\|w - z_\ell\|_W^2 + \|\lambda - \mu_{\mathscr{L}}\|_{-1/2}^2 + \Psi(q_\ell - p)\Big) + c_2\Big(b(\lambda - \mu_{\mathscr{L}}, \gamma(w) - \tilde{g}) \\
+ b(\lambda_{\mathscr{L}} - \mu, \gamma(w) - \tilde{g})\Big),
\end{aligned}$$

holds for all $z_\ell \in W_\ell$ and $\mu_{\mathscr{L}} \in \Lambda_{\mathscr{L}}$.

Proof. From the coercivity (1.11) we have

$$\nu_1\|w-w_\ell\|_W^2 \le a(w-w_\ell, w-w_\ell) = a(w-w_\ell, (u,0)-(u_\ell,0))+a(w-w_\ell, (0,p)-(0,p_\ell)). \tag{3.2}$$

For the second summand there holds

$$a(w-w_\ell,(0,p-p_\ell)) = a(w-w_\ell,(0,p-q_\ell)) + a(w-w_\ell,(0,q_\ell-p_\ell)).$$

And the choice $q = -q_\ell + 2p$ in Lemma 1.22 yields

$$\begin{aligned} a(w-w_\ell,(0,q_\ell-p_\ell)) &= a(w,(0,q_\ell-p)) + a(w,(0,p-p_\ell)) + a(w_\ell,(0,p_\ell-q_\ell)) \\ &\le \Psi(p-q_\ell) + \Psi(p_\ell) - \Psi(p) + \Psi(q_\ell) - \Psi(p_\ell) \\ &\le 2\Psi(q_\ell - p). \end{aligned}$$

Hence, we have

$$a(w-w_\ell,(0,p-p_\ell)) \le a(w-w_\ell,(0,p-q_\ell)) + 2\Psi(q_\ell-p).$$

In addition, the Lemma also implies

$$\begin{aligned} a(w-w_\ell,(u,0)-(u_\ell,0)) &= a(w-w_\ell,(u,0)-(v_\ell,0)) + a(w-w_\ell,(v_\ell,0)-(u_\ell,0)) \\ &= a(w-w_\ell,(u,0)-(v_\ell,0)) + b(\lambda_{\mathscr{L}}-\lambda,\gamma(v_\ell-u,0)) + b(\lambda_{\mathscr{L}}-\lambda,\gamma(u-u_\ell,0)). \end{aligned}$$

Furthermore, there holds

$$\begin{aligned} b(\lambda_{\mathscr{L}}-\lambda,\gamma(u-u_\ell,0)) &\le b(\mu-\lambda,\tilde{g}-\gamma(u_\ell,0)) + b(\lambda_{\mathscr{L}}-\mu,\gamma(u-u_\ell,0)) \\ &= b(\mu-\lambda_{\mathscr{L}},\tilde{g}-\gamma(u,0)) + b(\mu_{\mathscr{L}}-\lambda,\tilde{g}-\gamma(u_\ell,0)) \\ &\le b(\mu-\lambda_{\mathscr{L}},\tilde{g}-\gamma(u,0)) + b(\mu_{\mathscr{L}}-\lambda,\tilde{g}-\gamma(u,0)) + b(\mu_{\mathscr{L}}-\lambda,\gamma(u_\ell-u,0)), \end{aligned}$$

where we used the stationary conditions (2.7) and (1.40). Next, we put the estimates together, and use the continuity of a, functionals in $(H^{-1/2}(\Gamma_C))^d$ and the trace operator. This way, we arrive at

$$\begin{aligned} \nu_1\|w-w_\ell\|_W^2 \le{}& \nu_0\|w-w_\ell\|_W\|u-v_\ell\|_1 \\ &+ \|\gamma\|(\|\lambda-\lambda_{\mathscr{L}}\|_{-1/2}\|u-v_\ell\|_1 + \|\lambda-\mu_{\mathscr{L}}\|_{-1/2}\|u-u_\ell\|_1) \\ &+ b(\mu-\lambda_{\mathscr{L}},\tilde{g}-\gamma(u,0)) + b(\mu_{\mathscr{L}}-\lambda,\tilde{g}-\gamma(u,0)) + 2\Psi(q_\ell-p). \end{aligned}$$

Now, we use Lemma 3.5 and Young's inequality to conclude

$$\begin{aligned} \nu_0\|w-w_\ell\|_W^2 \le{}& \|w-w_\ell\|_W(\nu_1\|u-v_\ell\|_1 + \|\gamma\|\|\lambda-\mu_{\mathscr{L}}\|_{-1/2}) \\ &+ \|\lambda-\mu_{\mathscr{L}}\|_{-1/2}^2 + \|\gamma\|\|u-v_\ell\|_1^2 \\ &+ b(\mu-\lambda_{\mathscr{L}},\tilde{g}-\gamma(u,0)) + b(\mu_{\mathscr{L}}-\lambda,\tilde{g}-\gamma(u,0)) + 2\Psi(q_\ell-p). \end{aligned}$$

Finally, the assertion holds due to another application of Young's inequality. □

Together with Theorem 2.11, the above theorem also implies strong convergence of $\lambda_{\mathscr{L}}$ to λ if the density assumptions on the discrete space are fulfilled.

Remark 3.7. The term $b(\mu - \lambda_{\mathscr{L}}, \tilde{g} - \gamma(u, 0))$ measures the nonconformity of the discrete set of Lagrange multipliers. Note that if we have $\Lambda_{\mathscr{L}} \subset \Lambda$ the term becomes zero. This is for example possible for the choice of $\Lambda_{\mathscr{L}}$ as in the definitions (2.19) and (2.20), i.e., $\mathcal{M}_{\mathscr{L}}$ consists of piecewise constant, affine, bilinear, or linear functions.

If we assume an additional property of the discrete spaces we can use the orthogonality of the L^2-projection to proof the following estimate. This estimate will be useful when we derive convergence rates for the finite element discretization based on triangles which is known from e.g. [4].

Theorem 3.8. *Let the inf-sup condition* (2.12) *be fulfilled and* $\tilde{Q}_\ell := \varepsilon(V_\ell) \subseteq Q_\ell$. *Moreover, let* $\Pi_{\tilde{Q}_\ell} p$ *be the* L^2*-projection of* p *onto* $\tilde{Q}_\ell$. *There exist constants* $c_1, c_2 > 0$ *such that*

$$\|w - w_\ell\|_W^2 + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2}^2 \leq c_1\Big(\|w - z_\ell\|_W^2 + \|\lambda - \mu_{\mathscr{L}}\|_{-1/2}^2\Big) + c_2\Big(b(\lambda - \mu_{\mathscr{L}}, \gamma(w) - \tilde{g}) + b(\lambda_{\mathscr{L}} - \mu, \gamma(w) - \tilde{g}) + \Psi(\Pi_{\tilde{Q}_\ell} p) - \Psi(p)\Big),$$

holds for all $z_\ell \in W_\ell$ *and* $\mu_{\mathscr{L}} \in \Lambda_{\mathscr{L}}$.

Proof. The assertion holds by the same arguments as in the proof of Theorem 3.6. We only have to modify the estimation of the second summand in (3.2) as

$$\begin{aligned} a(w - w_\ell, (0,p) - (0,p_\ell)) &= a(w, (0,p) - (0,p_\ell)) + a(w_\ell, (0,p_\ell) - (0, \Pi_{\tilde{Q}_\ell} p)) \\ &\leq \Psi(p_\ell) - \Psi(p) + \Psi(\Pi_{\tilde{Q}_\ell} p) - \Psi(p_\ell) = \Psi(\Pi_{\tilde{Q}_\ell} p) - \Psi(p). \end{aligned}$$

Indeed, this holds due to Lemma 1.22 and the orthogonality of the L^2-projection. □

Next, we focus on the saddle point formulation (1.45) with three Lagrange multipliers. In a similar way as above, we use Lemma 3.2 to show an a priori result for the dual mixed formulation of Problem 2.13 with three Lagrange multipliers.

Theorem 3.9. *Let the inf-sup condition* (2.12) *be fulfilled and* $\Lambda_{P,\ell} \subset Q_\ell$. *There exist constants* $c_1, c_2 > 0$ *such that*

$$\begin{aligned} \|w - w_\ell\|_W^2 + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2}^2 + \|\lambda_P - \lambda_{P,\ell}\|_0^2 \leq \\ & c_1\Big(\|w - z_\ell\|_W^2 + \|\lambda - \mu_{\mathscr{L}}\|_{-1/2}^2 + \|\lambda_P - \mu_{P,\ell}\|_0^2\Big) \\ & + c_2\Big(b(\lambda - \mu_{\mathscr{L}}, \gamma(w) - \tilde{g}) + b(\lambda_{\mathscr{L}} - \mu, \gamma(w) - \tilde{g}) \\ & + (\lambda_P - \mu_{P,\ell}, \sigma_y p)_0 + (\lambda_{P,\ell} - \mu_P, \sigma_y p)_0\Big) \end{aligned}$$

holds for all $z_\ell \in W_\ell$, $\mu_{\mathscr{L}} \in \Lambda$, $\mu_{P,\ell} \in \Lambda_P$ *and* $\mu \in \Lambda$.

Proof. We use the variational equation in the stationary condition (1.45) and its discrete counterpart (2.13) to conclude that

$$
\begin{aligned}
a(w-w_\ell, w-w_\ell) &= a(w-w_\ell, w-z_\ell) + a(w-w_\ell, z_\ell-w_\ell) \\
&= a(w-w_\ell, w-z_\ell) + b(\lambda_{\mathscr{L}}-\lambda, \gamma(z_\ell-w)) + b(\lambda_{\mathscr{L}}-\lambda, \gamma(w-w_\ell)) \\
&\quad + (\lambda_{P,\ell}-\lambda_P, \sigma_y(q_\ell-p))_0 + (\lambda_{P,\ell}-\lambda_P, \sigma_y(p-p_\ell))_0.
\end{aligned}
$$

As in the proof of Theorem 3.6 we have

$$
\begin{aligned}
&b(\lambda_{\mathscr{L}}-\lambda, \gamma(w-w_\ell)) + (\lambda_{P,\ell}-\lambda_P, \sigma_y(p-p_\ell))_0 \le \\
&\qquad b(\mu-\lambda_{\mathscr{L}}, \tilde{g}-\gamma(w)) + b(\mu_{\mathscr{L}}-\lambda, \tilde{g}-\gamma(w)) + b(\mu_{\mathscr{L}}-\lambda, \gamma(w-w_\ell)) \\
&\qquad + (\lambda_{P,\ell}-\mu_P, \sigma_y p)_0 + (\lambda_P-\mu_{P,\ell}, \sigma_y p)_0 + (\lambda_P-\mu_{P,\ell}, \sigma_y(p-p_\ell))_0.
\end{aligned}
$$

The ellipticity and continuity of the bilinear form a, the continuity of functionals in $(H^{-1/2})^d$ and the Cauchy-Schwarz inequality lead to

$$
\begin{aligned}
\nu_0\|w-w_\ell\|_W^2 &\le \nu_1\|w-w_\ell\|_W\|w-z_\ell\|_1 + \|\gamma\|\|\lambda-\lambda_{\mathscr{L}}\|_{-1/2}\|w-z_\ell\|_1 \\
&\quad + \|\gamma\|\|\lambda-\mu_{\mathscr{L}}\|_{-1/2}\|w-w_\ell\|_1 + \sigma_y\|\lambda_P-\mu_{P,\ell}\|_0\|w-w_\ell\|_1 \\
&\quad + b(\mu-\lambda_{\mathscr{L}}, \tilde{g}-\gamma(w)) + b(\mu_{\mathscr{L}}-\lambda, \tilde{g}-\gamma(w)) \\
&\quad + (\lambda_{P,\ell}-\mu_P, \sigma_y p)_0 + (\lambda_P-\mu_{P,\ell}, \sigma_y p)_0.
\end{aligned}
$$

Finally, Lemma 3.5 and 3.2, as well as Young's inequality yield the assertion. □

Similar as for the problem with two Lagrange multipliers, we can combine Theorem 2.12 and the result just shown to conclude the strong convergence of the Lagrange multipliers.

Remark 3.10. If $\Lambda_{P,\ell} \not\subseteq Q_\ell$ we require the inf-sup condition 2.14 to be fulfilled instead and hence an analogon of Lemma 3.5 holds for the three Lagrange multipliers. The estimate of Theorem 3.9 is then shown by the same arguments as the ones used in the proof of Theorem 3.6.

3.2 A Priori Convergence Rates

In this section, we present convergence rates for the finite element discretizations of Section 2.3. The rates are formulated in terms of the mesh sizes and polynomial degrees of the discrete spaces for the primal variables and the Lagrange multipliers. In a standard approach, the rates arise from the combination of the results of the previous Section 3.1 with standard interpolation estimates. In order to use the interpolation operators and error estimates we assume sufficient regularity of the analytic solution.

Throughout this section, we assume

$$u \in H^{1+\theta}(\Omega, \mathbb{R}^d),\ p \in H^{\tilde{\theta}}(\Omega, \mathbb{R}^{d\times d}),\ \text{and}\ \lambda \in H^{\bar{\theta}}(\Gamma_C, \mathbb{R}^d).$$

Moreover, equation (1.46) expresses the Lagrange multiplier λ_P in terms of displacement u and plastic strain p. Let $\hat{\theta} = \min(\theta, \tilde{\theta})$. Hence, the above regularity assumptions yield $\lambda_P \in H^{\hat{\theta}}(\Omega, \mathbb{R}^{d\times d})$.

Let $\mathbb{H} = \xi\mathbb{I}$ which is sometimes referred to as holonomic elastoplasticity [84, 83, 55]. Furthermore, let $\lambda, \mu > 0$ be such that the tensor $\mathbb{C}$ of linear elasticity can be written as $\mathbb{C}\tau = 2\mu_{\mathbb{C}}\tau + \lambda_{\mathbb{C}} \operatorname{tr}(\tau)\mathbb{I}$ for all $\tau \in \mathbb{R}^{d\times d}$. The material constants $\lambda_{\mathbb{C}}$ and $\mu_{\mathbb{C}}$ are called Lamé's first and second parameter, respectively. In this setting, the regularity of the plastic strain and the displacement are linked for the problem 1.14 with linear kinematic hardening and without contact. The minimization of the energy functional E_P with respect to the plastic strain p yields this link, c.f. [4]. The result is an explicit expression of p in terms of the displacement

$$p = \frac{(|\operatorname{dev}(\mathbb{C}\varepsilon(u))| - \sigma_y)_+}{2\mu_{\mathbb{C}} + \xi} \frac{\operatorname{dev}(\mathbb{C}\varepsilon(u))}{|\operatorname{dev}(\mathbb{C}\varepsilon(u))|}.$$

Moreover, note that the regularity of λ_P and w are coupled via the identity (1.46). We observe that the minimization of E with respect to p is the same minimization problem as for E_P, since the contact conditions have no direct effect on the plastic strain. Hence the result still holds for the problem with contact conditions.

For the isotropic hardening model a similar result holds. In [53, Eq. (3.23)], the identity

$$p = \frac{(|\operatorname{dev}(\mathbb{C}\varepsilon(u))| - \sigma_y)_+}{2\mu_{\mathbb{C}} + \sigma_y^2 H^2} \frac{\operatorname{dev}(\mathbb{C}\varepsilon(u))}{|\operatorname{dev}(\mathbb{C}\varepsilon(u))|}$$

is shown.

In view of Theorems 3.6, 3.8, and 3.9, we want to choose z_h, μ_H and μ in such a way that we can specify the upper bounds in terms of mesh size, polynomial degree of the discrete solution and the regularity of the analytical solution. To this purpose, we will make use of the following standard interpolation results, cf. [9, 14]. Let $I_H^m : H^{1/2}(\Gamma_C) \to M_H^m$ and $I_h^l : H^1(\Omega) \to V_h^l$ be the piecewise interpolation operators using the sets of transformed Gauss points $\mathcal{G}^{d-1}$ and Gauss-Lobatto points, respectively. Furthermore, Let $\mu \in H^{\bar{\theta}}(\Gamma_C)$ with $\bar{\theta} > (d-1)/2$ and $v \in H^{1+\theta}(\Omega)$ with $\theta > d/2$. It holds

$$\|\mu - I_H^m \mu\|_{0,\Gamma_C} \lesssim H^{\min(m+1,\bar{\theta})} / \max(1,m)^{\bar{\theta}} \|\mu\|_{\bar{\theta}} \tag{3.3}$$

$$\|v - I_h^l v\|_1 \lesssim h^{\min(l,\theta)} / \max(1,l)^{\theta} \|v\|_{1+\theta}. \tag{3.4}$$

Moreover, let Π_{Q_ℓ} be the usual L^2 projection onto the $Q_\ell := Q_h^k$. The estimate

$$\|q - \Pi_{Q_\ell} q\|_0 \lesssim h^{\min(k+1,\tilde{\theta})} / (k+1)^{\tilde{\theta}} \|q\|_{\tilde{\theta}} \tag{3.5}$$

holds for all $q \in H^{\tilde{\theta}}(\Omega)$, see [86, 13]

3.2.1 Elastoplasticity

For the case where contact and friction conditions are absent, we repeat a convergence rate result from [97]. The result holds if we use the piecewise constant, affine, bilinear or trilinear finite element spaces of Section 2.3.

Corollary 3.11 ([97, Corollary 8.1])**.** *Let* $W_\ell := W_h^{l,k}$ *with* $k = 0, 1$ *and* $\hat{\theta} = \min\{\theta, \tilde{\theta}\}$*. Hence, there holds*

$$\|w - w_\ell\| + \|\lambda_P - \lambda_{P,\ell}\|_0 \lesssim h^{\min\{1,\hat{\theta}\}}.$$

In [93], Schröder shows how to derive an a priori result for discretizations of the mixed formulation of the problem with prescribed frictional force based on polynomial degrees greater than one. However, the arguments used there only hold for two spatial dimensions since, in this case, the tangential displacement reduces to a scalar quantity. In the following, we present the attempt to directly transfer the idea to the discretization of the saddle point problem (1.17), and show why this is not possible.

The starting point is the estimate

$$(\lambda_{P,h}^k - \mu, p)_0 \le |(\lambda_{P,h}^k - \mu, p - I_h^k(p))_0| + (\lambda_{P,h}^k - \mu, I_h^k(p))_0.$$

Now, the idea in [93] consists of the definition of a $\delta \in \Lambda_P \cap Q_\ell$ such that

$$(\lambda_{P,h}^k - \delta, I_h^k(p))_0 = \sum_\xi \lambda_{P,h}^k(\xi) - \delta(\xi)p(\xi) \le 0.$$

Note that the condition $\delta \in \Lambda_P$ is mandatory since we want to replace μ whereas $\delta \in Q_\ell$ stems from the wish to use the exactness of the Gaussian quadrature rule for polynomials.

For the scalar frictional side condition, it is easy to guarantee that $\lambda_{f,H}^m - \delta$ is either non positive or non negative, or $\gamma_t(u) = 0$ in all points of the Gaussian quadrature rule for a single edge. This is possible if the mesh width is sufficiently small due to the continuity of the analytic Lagrange multiplier. We will give a detailed proof of this in the next section. However, $\lambda_{P,h}^k$ is vector valued and therefore $\lambda_{P,h}^k(\xi) - \delta(\xi)p(\xi) \le 0$ means that $\lambda_{P,h}^k(\xi) - \delta(\xi)$ points away from $p(\xi)$ in the sense that the angle between the two is greater than π or less than $-\pi$, respectively. Since we can not assume a coupling between the directions of $\lambda_{P,\ell}(\xi)$ and $p(\xi)$ we consider the worst case $\lambda_{P,\ell}(\xi) = cp(\xi)$ with $c > 0$, i.e. $\lambda_{P,\ell}(\xi)$ points in the same direction as $p(\xi)$. Thus, we have to choose $\delta(\xi) = -\lambda_{P,\ell}(\xi)$. For another Gauss point $\tilde{\xi}$ in the same cell and independent of the mesh width, the direction of $p(\tilde{\xi})$ can change without becoming zero. Again, we consider the fact that $\lambda_{P,\ell}(\tilde{\xi})$ can have the same direction as $p(\tilde{\xi})$ and therefore the only possible choice for δ would be $\delta(\tilde{\xi}) = -\lambda_{P,\ell}(\tilde{\xi})$. Hence, in the worst case we have to choose $\delta = -\lambda_{P,\ell}$ and therefore can not guarantee $\delta \in \Lambda_P$. The same arguments show that this approach is not suitable for problems with two dimensional frictional restrictions.

3.2.2 Frictional Contact

If the discrete set of Lagrange multipliers $\Lambda_{\mathscr{L}}$ consists of piecewise constant or affine functions we have $\Lambda_{\mathscr{L}} \subset \Lambda$ and therefore $b(\lambda_{\mathscr{L}} - \mu, \gamma(w) - \tilde{g}) = 0$. Hence, we can easily deduce the following result from Theorem 3.6.

Theorem 3.12. *Let $W_\ell = W_h^{l,k}$, $\Lambda_{\mathscr{L}} = \Lambda_{C,H}^m \times \Lambda_{F,H}^m$ with $m = 0,1$ and $l \geq 1$. Moreover, let the inf-sup condition (2.12) be fulfilled. For the solution of (1.39) $w \in H^{1+\theta}(\Omega) \times H^{\tilde{\theta}}(\Omega)$ with $\theta, \tilde{\theta} > 0$, $\gamma(w) - \tilde{g} \in H^{\theta_C}(\Gamma_C)$, and the Lagrange multiplier $\lambda \in (H^{\bar{\theta}}(\Gamma_C))^d$ with $\bar{\theta}, \theta_C > 1/2$, it holds*

$$\begin{aligned}\|w - w_\ell\|_W + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2} \lesssim & h^{\min(l,\theta)}/l^\theta + h^{\min(k+1,\tilde{\theta})}/(k+1)^{\tilde{\theta}} + H^{\min(m+1,\bar{\theta})+1/2} \\ & + H^{(\min(m+1,\bar{\theta})+\min(m+1,\theta_C))/2} \\ & + h^{\min(k+1,\bar{\theta})/2}/(k+1)^{\bar{\theta}/2}.\end{aligned}$$

Proof. Note that $m = 0,1$ implies $\Pi_{M_H^m}(\lambda) \in \Lambda$. From the estimate (3.5) of the error of the L^2-projection as well as its orthogonality, we conclude

$$\begin{aligned}\|\lambda - \Pi_{M_H^m}(\lambda)\|_{-1/2} &= \sup_{z \in H^{1/2}(\Gamma_C), \|z\|_{1/2}=1} \int_{\Gamma_C} (\lambda - \Pi_{M_H^m}(\lambda))(z - \Pi_{M_H^m}(z)) \\ &\lesssim H^{\min(m+1,\bar{\theta})+1/2}/m^{\bar{\theta}}.\end{aligned}$$

Next, we estimate

$$\begin{aligned}b(\lambda - \Pi_{(M_H^m)^d}(\lambda), \gamma_n(w) - g) &= b(\lambda - \Pi_{(M_H^m)^d}(\lambda), \gamma_n(w) - g - \Pi_{(M_H^m)^d}(\gamma_n(w) - g)) \\ &\lesssim H^{(\min(m+1,\bar{\theta})+\min(m+1,\theta_C))}/m^{\bar{\theta}+\theta_C}\end{aligned}$$

where we used Cauchy's inequality, the interpolation result (3.3) and the orthogonality of the L^2-projection. Furthermore, we have

$$\Psi(\Pi_{Q_h^k}(p) - p) \lesssim \|\Pi_{Q_h^k}(p) - p\|_0 \lesssim h^{\min(k+1,\bar{\theta})}/k^{\bar{\theta}}$$

by Hölder's inequality. Note that $b(\lambda_{\mathscr{L}} - \mu, \gamma(w) - \tilde{g}) = 0$. We conclude the proof by the application of this, the three inequalities and the interpolation results to the estimate of Theorem 3.6. □

Note that it remains unclear under which conditions the regularity assumptions made in the previous theorem hold. Hence, the choice of the optimal polynomial degrees is not clear in general. The next Lemma allows us to drop the last term of the estimate for meshes consisting of triangles.

Lemma 3.13. *If the mesh only consist of triangles and V_ℓ consists of piecewise affine and Q_ℓ of piecewise constant functions the term $\Psi(\Pi_{Q_\ell} p) - \Psi(p)$ is non-positive.*

Proof. We follow the ideas in [4] and use Jensen's inequality to conclude

$$\int_T \sigma_y \left| \int_T \frac{p}{|T|} \right| - \int_T \sigma_y |p| = \left| \int_T \sigma_y p \right| - \int_T \sigma_y |p| \leq 0$$

on all triangles T. □

Hence, for a mesh consisting only of triangles, and for piecewise constant functions in the discretization of the plastic strain we have

$$\begin{aligned}\|w - w_\ell\|_W + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2} \lesssim & h^{\min(1,\theta)} + h^{\min(1,\tilde{\theta})} + H^{\min(1,\bar{\theta})+1/2} \\ & + H^{(\min(1,\bar{\theta})+\min(1,\theta_C))/2}.\end{aligned}$$

This estimate corresponds to the combination of estimates known for the problems of elastoplasticity with linear kinematic hardening [57, 26] and mixed formulations for Signorini's problem in linear elasticity [60].

However, when using quadrilaterals and piecewise bilinear functions for V_ℓ the choice of Q_ℓ as piecewise constant functions leads to $Q_\ell \subsetneq \operatorname{dev}(\sigma(V_\ell))$. But in a fashion similar to Theorem 3.12, we can use Theorem 3.9 to deduce convergence rates for the discrete saddle point problem (1.45) in the case that the Lagrange multiplier concerning the plastic dissipation functional is piecewise constant, linear, bilinear or trilinear.

Theorem 3.14. *Let $W_\ell = W_h^{l,k}$, $\Lambda_{\mathscr{L}} = \Lambda_{C,H}^m \times \Lambda_{F,H}^m$ and $\Lambda_{P,\ell} = \Lambda_{P,h}^k$ with $k, m \in \{0,1\}$ and $l \geq 1$. Moreover, let the inf-sup condition* (2.12) *be fulfilled and $\Lambda_{P,\ell} \subset Q_\ell$. For the solution of* (1.4) *$w \in H^{1+\theta}(\Omega) \times H^{\tilde{\theta}}(\Omega)$ with $\theta, \tilde{\theta} > 0$, $\gamma(w) - \tilde{g} \in H^{\theta_C}(\Gamma_C)$, and the Lagrange multiplier $\lambda \in (H^{\bar{\theta}}(\Gamma_C))^d$ with $\bar{\theta}, \theta_C > 1/2$ as well as $\lambda_P \in H^{\hat{\theta}}(\Omega)$ with $\hat{\theta} = \min(\theta, \tilde{\theta})$, it holds*

$$\begin{aligned}\|w - w_\ell\|_W + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2} \lesssim & h^{\min(l,\theta)}/l^\theta + h^{\min(k+1,\tilde{\theta})}/(k+1)^{\tilde{\theta}} + H^{\min(m+1,\bar{\theta})+1/2} \\ & + H^{(\min(m+1,\bar{\theta})+\min(m+1,\theta_C))/2} \\ & + h^{(\min(k+1,\hat{\theta})+\min(k+1,\bar{\theta}))/2}/(k+1)^{(\hat{\theta}+\bar{\theta})/2}.\end{aligned}$$

Proof. From $\Lambda_{P,\ell} \subset \Lambda_P$, we observe $(\lambda_{P,\ell} - \mu_P, \sigma_y p)_0 = 0$. Moreover,

$$(\lambda_P - \Pi_{Q_\ell}(p), \sigma_y p)_0 = (\lambda_P - \mu_{P,\ell}, \sigma_y (p - \Pi_{Q_\ell}(p)))_0 \leq h^{(\min(k+1,\hat{\theta})+\min(k+1,\tilde{\theta}))}/(k+1)^{(\hat{\theta}+\tilde{\theta})}.$$

Now, this and the similar arguments as in the proof of the previous theorem yield the assertion. □

For polynomial degrees greater then one, it is not obvious how to choose μ in such a way that $b(\lambda_{\mathscr{L}} - \mu, \gamma(w) - \tilde{g}) = 0$. However, in [99, 93] it is shown that for $d = 2$ it is possible to obtain an estimate on the rate of convergence for the term

$b(\lambda_{\mathscr{L}} - \mu, \gamma(w) - \tilde{g})$ as well. To show this we make use of the inverse inequality

$$\|\mu_{C,H}^m\|_{-1/2+\tilde{\theta}} \lesssim \max(1, m^{2\tilde{\theta}})/H^{\tilde{\theta}} \|\mu_{C,H}^m\|_{-1/2} \tag{3.6}$$

which holds for all $\mu_{C,H}^m \in M_H^m$. For a proof we refer to [51].

Lemma 3.15. *Let* $\Omega \subset \mathbb{R}^2$ *and the inf-sup condition* (2.12) *be fulfilled. Moreover, let* $f_F \gamma_t(w) \in H^{\theta_F}$ *change its sign only in a finite number of points and* $\gamma_n(w) - g \in H^{\theta_C}$. *Then, there holds*

$$b(\lambda_H^m - \mu, \gamma(w) - \hat{g}) \lesssim H^{\min(m+1,\theta_C)-1/2}/(m)^{\theta_C - 1} + H^{\min(m+1,\theta_F)}/(m)^{\theta_F}$$

for H *sufficiently small.*

Proof. We follow the arguments of Schröder [93]. At first, we observe that the sequence λ_H^m is weakly convergent due to Theorem 2.11 and therefore bounded. Furthermore, we note that

$$\langle \lambda_{C,H}^m, \gamma_n(w) - g \rangle = (\lambda_{C,H}^m, \gamma_n(w) - g)_{0,\Gamma_C}$$

since $\lambda_{C,H}^m \in M_h^m$. Now, we start with the estimate for the term with the multiplier $\lambda_{C,H}^m$ which captures the contact condition. We recall that $\lambda_{C,H}^m(\xi) \geq 0$ for all $\xi \in \mathcal{G}$ due to the definition of Λ_C and $(\gamma_n(w) - g)(x) \leq 0$ for almost all $x \in \Gamma_C$ which is the non-penetration condition (1.33). Let α_ξ denote the weights of the Gaussian quadrature rule associated to the nodes ξ. We recall that these weights are all positive. Together with the properties of the Gaussian quadrature for the integration of polynomials, we conclude

$$(\lambda_{C,H}^m, I_H^m(\gamma_n(w) - g))_{0,\Gamma_C} = \sum_{\xi \in \mathcal{G}} \alpha_\xi \lambda_{C,H}^m(\xi)(\gamma_n(w) - g)(\xi) \leq 0.$$

Let $\mu_C = 0$. With the inverse inequality (3.6), we have

$$\begin{aligned} (\lambda_{C,H}^m, \gamma_n(w) - g)_{0,\Gamma_C} &\leq |(\lambda_{C,H}^m, \gamma_n(w) - g - I_H^m(\gamma_n(w) - g))_{0,\Gamma_C}| \\ &\quad + (\lambda_{C,H}^m, I_H^m(\gamma_n(w) - g))_{0,\Gamma_C} \\ &\leq \|\lambda_{C,H}^m\|_{0,\Gamma_C} \|\gamma_n(w) - g - I_H^m(\gamma_n(w) - g)\|_{0,\Gamma_C} \\ &\lesssim H^{\min(m+1,\theta_C)-1/2}/(m)^{\theta_C - 1}. \end{aligned}$$

Next, we estimate the second term in $b(\lambda_H^m - \mu, \gamma(w) - \hat{g})$ which incorporates the multiplier $\lambda_{F,H}^m$ of the friction condition. We define

$$\mathcal{E}_H^* := \{E \in \mathcal{E}_H \mid \sup_{x \in E} |\lambda_F(x)| < 1\} \quad \text{and} \quad \mathcal{E}_H^{\pm} := \{E \in \mathcal{E}_H \setminus \mathcal{E}_H^* \mid \inf_{x \in E} \pm\lambda_F(x) \geq 0\}.$$

We have $(\lambda_{F,H}^m(\xi) \pm 1)\gamma_t(w)(\xi) \leq 0$ for all $\xi \in \mathcal{G}$ with $\xi \in E \in \mathcal{E}_H^{\pm}$ since $\pm\gamma_t(w)(\xi) \leq 0$. Additionally, $\gamma_t(w)_{|E} = 0$ for $E \in \mathcal{E}_H^*$ and $|\lambda_{F,H}^m(\xi)| \leq 1$ for $\xi \in \mathcal{G}$. This in turn is a direct consequence of the friction condition (1.34) and the identity of λ and σ_t. Since λ_F is continuous we have $\mathcal{E}_H = \mathcal{E}_H^* \cup \mathcal{E}_H^+ \cup \mathcal{E}_H^-$ for $H = \max_E |E|$ sufficiently small. Furthermore, we set $\delta_{|E} := \pm 1$ for $E \in \mathcal{E}_H^{\pm}$ and 0 elsewhere. Note that $\delta \in \Lambda$ and $\delta \in M_H^m$. We choose $\mu_F = \delta$ and observe

$$(\lambda_{F,H}^m - \delta, I_H^m(\gamma_t(w)))_{0,\Gamma_C} = \sum_{\xi} \lambda_{F,H}^m(\xi) - \delta(\xi)(\gamma_t(w))(\xi) \leq 0.$$

We are now able to conclude

$$\begin{aligned}(\lambda_{F,H}^m - \delta, \tilde{f}_F \gamma_t(w))_{0,\Gamma_C} &\leq |(\lambda_{F,H}^m - \delta, \tilde{f}_F \gamma_t(w) - I_H^m(\tilde{f}_F \gamma_t(w)))_{0,\Gamma_C}| \\ &\quad + (\lambda_{F,H}^m - \delta, I_H^m(\tilde{f}_F \gamma_t(w)))_{0,\Gamma_C} \\ &\leq \|\lambda_{F,H}^m - \delta\|_{0,\Gamma_C} \|\tilde{f}_F \gamma_t(w) - I_H^m(\tilde{f}_F \gamma_t(w))\|_{0,\Gamma_C} \\ &\lesssim H^{\min(m+1,\theta_F)}/(m)^{\theta_F}.\end{aligned}$$

Which together with the first estimate yield the assertion of the Lemma. □

Altogether, for the discretization with polynomial degrees bigger than one, we get the following convergence rates for the sequence of the discrete saddle points $(w_\ell, \lambda_{\mathscr{L}})$ of $\mathcal{L}_{C,F}$.

Theorem 3.16. *Let the hold the assumptions of Lemma 3.15 $\Omega \subset \mathbb{R}^2$ and $l, m \geq 1$. Moreover, let the solution $w \in H^\theta(\Omega) \times H^{\bar{\theta}}(\Omega)$, $\gamma(w) - \tilde{g} \in H^{\hat{\theta}}(\Gamma_C)$ and the Lagrange multiplier $\lambda \in (H^{\tilde{\theta}}(\Gamma_C))^d$ with $\theta > 1$, $\bar{\theta} > 0$ and $\theta_C, \tilde{\theta} > 1/2$. Then, there holds*

$$\begin{aligned}\|w - w_\ell\|_W + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2} \lesssim & h^{\min(l,\theta)}/l^\theta + h^{\min(k+1,\bar{\theta})}/(k+1)^{\bar{\theta}} + h^{\min(k+1,\bar{\theta})/2}/(k+1)^{\bar{\theta}} \\ & + H^{(\min(m+1,\theta_C)-1/2)/2}/m^{(\theta_C-1)/2} \\ & + H^{\min(m+1,\theta_C)/2}/m^{\theta_C/2} + H^{\min(m+1,\tilde{\theta})/2}/m^{\tilde{\theta}}\end{aligned}$$

Proof. Theorem 3.6, Lemma 3.15, the interpolation estimates and

$$\begin{aligned}\|\lambda - I_H^m(\lambda)\|_{-1/2} &= \sup_{z \in H^{1/2}(\Gamma_C), \|z\|_{1/2;\Gamma_C}=1} (\lambda - I_H^m(\lambda), z)_{0,\Gamma_C} \\ &\leq \sup_{z \in L^2, \|z\|_{0,\Gamma_C}=1} \|\lambda - I_H^m(\lambda)\|_{0,\Gamma_C} \|z\|_{0,\Gamma_C} \\ &\lesssim H^{\min(m+1,\tilde{\theta})}/(m)^{\tilde{\theta}}\end{aligned}$$

directly yield the assertion. □

It would be desirable to also establish such a priori convergence rates for the discretization with three Lagrange multipliers of polynomial degrees greater than one or for problems in three spatial dimensions. However, the proof of Lemma 3.15 relies strongly on the fact that for $d = 2$ the Lagrange multiplier for the friction functional is a real valued function. Since this no longer given if $d = 2$ or for the

Lagrange multiplier for the plastic dissipation functional the argumentation of the Lemma can not be transferred directly.

4 A posteriori estimates

In contrast to the a priori estimates of the previous chapter, a posteriori error estimates are based on quantities computable from the discrete solution. The estimation of the discretization error for linear elliptic problems can easily be done via some generic approaches as presented for example in [2, 106, 31, 30]. Moreover, a variety of estimates is known for Signorini's problem of linear elasticity as well as for the primal problem of elastoplasticity with hardening. These estimates reach from residual [4] over equilibrated [39], averaging [29], dissipation error [75] and functional estimators [85] up to dual weighted residual approaches [81]. In [113], Zarrabi presents a residual based a posteriori error estimator for a two body contact problem in elastoplasticity.

Furthermore, their localization is used as a local refinement indicator. Provided we use affine and constant basis functions on triangles and a marking of Dörfler type [47], this results in an adaptive refinement scheme shown to be optimal for linear elliptic problems with for example conforming low order [42] or even mixed methods [36].

An a posteriori estimate for the mixed formulation of the Signorini problem based on low order finite elements can be found in [89, 90]. However, the discretization with Lagrange multipliers of polynomial degrees greater than one results in non conform schemes. Nevertheless, a posteriori results are presented in [94].

4.1 Residual error estimator

In this section, we introduce residual error estimators for the discretization with and without Lagrange multiplier. First, we consider the problem of elastoplasticity with hardening and without contact conditions. This is followed by some residual estimates for the elastoplastic problem with contact, i.e., the discrete saddle points of $\mathcal{L}_{CF}$ and $\mathcal{L}$, respectively. The estimators can be derived by standard arguments as used for linear elliptic problems [106, 22]. We use the residual estimators to control the discretization error in some norm up to a generic constant.

We derive residual error estimates for all spaces which were introduced in Section 2.3. However, we give special attention to the a posteriori bounds for the elastoplastic problem without contact conditions on triangular meshes well known from [4]. Moreover, we especially highlight a result for an approach based on quadrilateral cells. Since we will be able to show convergence for adaptive finite element schemes based on these approaches in the subsequent chapter.

4.1.1 Elastoplasticity

As mentioned above, the discrete variational inequality of Definition 2.1 is equivalent to the discrete saddle point problem for the approximation based on a piecewise affine or bilinear ansatz for the displacement and piecewise constant ansatz for the plastic strain. For this discretization of elastoplasticity without contact conditions, residual estimators can be found for example in [4, 19, 97]. We briefly repeat some of the well known results. We set

$$\eta_{P,\ell}^2(T) := |T| R_{P,T}^2 + |T|^{1/2} \sum_{E \in \mathcal{E}(T)} R_{P,E}^2$$

with

$$R_{P,T} := \|f_\Omega\|_{0,T} \text{ and } R_{P,E} := \begin{cases} \|[\sigma(w_\ell)]\nu_E\|_{L^2(E;\mathbb{R}^d)} & \text{for } E \in \mathcal{E}_\ell^\circ, \\ \|f_N - \sigma(w_\ell)\nu_E\|_{L^2(E;\mathbb{R}^d)} & \text{for } E \in \mathcal{E}_\ell^N, \\ 0 & \text{for } E \subset \Gamma_D. \end{cases}$$

Moreover, we introduce the norm $\|\sigma\|_{\mathbb{C}^{-1}}^2 := (\sigma, \mathbb{C}^{-1}\sigma)_0$. The oscillations on a subset $\mathcal{M} \subset \mathcal{T}_\ell$ are defined in the usual way by

$$\operatorname{osc}^2(f_\Omega, K) := |K| \|f_\Omega - f_K\|_{0,K}^2 \tag{4.1}$$

$$\operatorname{osc}^2(f_\Omega, \mathcal{M}) := \sum_{K \in \mathcal{M}} \operatorname{osc}^2(f_\Omega, K). \tag{4.2}$$

Furthermore, on a subset $\mathcal{F} \subset \mathcal{E}_\ell^N$ of edges or faces the oscillations read

$$\operatorname{osc}^2(f_N, E) := |E|^{1/2} \|f_N - f_E\|_{0,E}^2 \tag{4.3}$$

$$\operatorname{osc}^2(f_N, \mathcal{F}) := \sum_{E \in \mathcal{F}} \operatorname{osc}^2(f_N, E). \tag{4.4}$$

Theorem 4.1. *Let $\mathcal{T}_\ell$ consist of only triangles and $W_\ell := V_h^1 \times Q_h^0$. The estimator*

$$\eta_{P,\ell}^2 := \sum_{T \in \mathcal{T}_\ell} \eta_{P,\ell}^2(T)$$

is efficient and reliable in the sense that

$$\|\sigma(w) - \sigma(w_\ell)\|_{\mathbb{C}^{-1}} \lesssim \eta_{P,\ell} \lesssim \|\sigma(w) - \sigma(w_\ell)\|_{\mathbb{C}^{-1}} + \operatorname{osc}(f_\Omega, \mathcal{T}_\ell) + \operatorname{osc}(f_N, \mathcal{E}_\ell^N).$$

Proof. See proofs of inequalities (41), (47) and (48) in [40]. □

The proof of this result explicitly uses the discretization by affine and constant functions and does not hold for polynomials of higher degrees nor quadrilateral meshes.

Next, we want to derive an error estimator for bilinear finite elements based on quadrilaterals. To this end, we focus on the saddle point formulation of Definition 2.2 and polynomial degrees not greater than one. In this setting, we than have $\Lambda_{P,\ell} \subset \Lambda$ and w_ℓ is also the solution of the variational inequality of Definition 2.1, cf. [97]. Hence, we will see how the estimates on the error $w - w_\ell$ still hold for a direct approach without Lagrange multipliers. We define the residual

$$\langle \mathrm{Res}_P(w_\ell), v\rangle := (f, v)_0 + (f_N, \gamma(v))_{0,\Gamma_N} - (\sigma(w_\ell), \varepsilon(v))_0$$

and cite the following result.

Lemma 4.2 ([97, Lemma 6.1])**.** *If $\Lambda_{P,\ell} \subset \Lambda_P$ holds then*

$$\|w - w_\ell\|_W \lesssim \|\operatorname{Res}_P(w_\ell)\|_{W'} + \|\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y \lambda_{P,\ell}\|_0.$$

The residual is defined the same way as if the triangulation would only include triangles. Furthermore, the usual Clément interpolation results still hold for meshes with hanging nodes, see [34]. We observe

$$\|\operatorname{Res}_P(w_\ell)\|^2_{W'} \lesssim \eta^2_{P,\ell} \lesssim \|\operatorname{Res}_P(w_\ell)\|^2_{W'} + \mathrm{osc}^2$$

by standard arguments [107]. Therefore, the estimator $\eta_{P,\ell}$ of Theorem 4.1 still can be used to define an upper bound for the error. In fact, the error can be bounded by the sum of the estimator η_ℓ and the deviatoric term.

If we want to use the estimate in a setting without Lagrange multipliers we have to specify how $\lambda_{P,\ell}$ can be eliminated. Note that the mixed and the direct discrete approach are equivalent if $\Lambda_{P,\ell} \subset \Lambda_P$. By the same arguments as for the analytical Lagrange multiplier, we can show that the term $\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y \lambda_\ell$ equals zero under some additional assumptions.

Lemma 4.3. *If for all $z_\ell \in W_\ell$ holds $\operatorname{dev}(\sigma(z_\ell)) \in Q_\ell$ then*

$$\|\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y \lambda_{P,\ell}\|_0 = 0.$$

In other words

$$\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell)/\sigma_y = \lambda_{P,\ell}.$$

Proof. Similar as for the determination of the analytic Lagrange multiplier, we choose $z_\ell = (0, q_\ell)$ with arbitrary $q_\ell \in Q_\ell$ and conclude

$$\begin{aligned} 0 &= -a(w_\ell, z_\ell) - (\lambda_{P,\ell}, \sigma_y q_\ell)_0 \\ &= (\sigma(w_\ell) - \mathbb{H}p_\ell, q_\ell)_0 - (\lambda_{P,\ell}, \sigma_y q_\ell)_0 \\ &= (\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y \lambda_{P,\ell}, q_\ell)_0 \end{aligned}$$

and $\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y \lambda_{P,\ell} \in Q_\ell$ shows the assertion. □

Additionally, the proof of Lemma 4.3 shows that $\lambda_{P,\ell}$ is the L^2-projection of the term $\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell)/\sigma_y$ onto Q_ℓ regardless of the choice of the discretization spaces. We conclude that the error $w - w_\ell$ of a direct discretization approach for the variational inequality without Lagrange multipliers is bounded from above by the sum of the norms of the residual and the L^2-projection error of the deviatoric term, i.e.,

$$\|w - w_\ell\|_W^2 \lesssim \|\operatorname{Res}_P(w_\ell)\|_{W'}^2 + \|(\mathbb{I} - \Pi_{Q_\ell})(\operatorname{dev}(\sigma(w_\ell))/\sigma_y)\|_0^2.$$

Hence, we have

$$\|w - w_\ell\|_W^2 \lesssim \eta_{P,\ell}^2 + \|(\mathbb{I} - \Pi_{Q_\ell})(\operatorname{dev}(\sigma(w_\ell))/\sigma_y)\|_0^2.$$

Note that $(\mathbb{I} - \Pi_{Q_\ell})(\mathbb{H}p_\ell) = 0$.

Additionally, if $\Lambda_{P,\ell} \subset \Lambda_P$ the estimator is even efficient up to oscillations, i.e.,

$$\eta_{P,\ell} + \|(\mathbb{I} - \Pi_{Q_\ell})(\operatorname{dev}(\sigma(w_\ell))/\sigma_y)\|_0 \lesssim \|w - w_\ell\|_W^2 + \operatorname{osc}_\ell^2 .$$

This directly follows from standard arguments known for the residual error estimation in linear elasticity as found for example in [107].

Note that the deviatoric term is not scaled by the element size. Thus, the usual arguments to show an estimator reduction and consequently the convergence of the adaptive finite element method do not apply. However, if the assumptions of Lemma 4.3 hold and thus the deviatoric term is zero we have the same estimator as for triangulations based on triangles. In this case, it is possible to prove convergence of the adaptive scheme.

Next, we focus on the saddle point formulation with polynomials of arbitrary degree. We use the analogy of some aspects of this formulation with the discrete mixed formulation of the Tresca friction problem in linear elasticity. This makes it easy to use the arguments presented in [94] to derive an a posteriori estimate for the elastoplastic problem with hardening. We begin the derivation with the introduction of the unique solution w_* of the variational equality

$$a(w_*, z) = \mathcal{F}(z) - (\lambda_{P,\ell}, \sigma_y q)_0 \tag{4.5}$$

for all $z \in W$. The discretization of this equality with W_ℓ obviously defines the same discrete solution w_ℓ as the first component of the saddle point. Since the bilinear form a is W-elliptic we use the Clément interpolant of the discretization error in the displacement and the Cauchy-Schwarz inequality to conclude

$$\|w_* - w_\ell\|_W^2 \lesssim \eta_{P,\ell}^2 + \|\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y\lambda_{P,\ell}\|_0^2. \tag{4.6}$$

We can use the estimate (4.6) to bound the error of the primal variable.

Theorem 4.4. *Let w, λ_P be the solution of* (1.16) *and $w_\ell, \lambda_{P,\ell}$ be the solution of*

(2.3),(2.4). *The error of the primal solution can be estimated as follows*

$$\begin{aligned}\|w - w_\ell\|^2 \lesssim \eta_{P,\ell}^2 &+ \|\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y\lambda_{P,\ell}\|_0^2 \\ &+ \|\lambda_{P,\ell} - \mu_P\|_0^2 + \Psi(p_\ell) - (\mu_P, \sigma_y p_\ell)_0\end{aligned}$$

with arbitrary $\mu_P \in \Lambda_P$.

Proof. We have

$$\begin{aligned}\nu_0\|w - w_\ell\|^2 &\le a(w - w_\ell, w - w_\ell) = a(w - w_*, w - w_\ell) + a(w_* - w_\ell, w - w_\ell) \\ &\le (\lambda_{P,\ell} - \lambda_P, \sigma_y(p - p_\ell))_0 + \nu_1\|w_* - w_\ell\|\|w - w_\ell\| \\ &\le (\lambda_{P,\ell} - \mu_P, \sigma_y(p - p_\ell))_0 + (\mu_P - \lambda_P, \sigma_y(p - p_\ell))_0 \\ &\quad + \nu_1(\delta^{-1}\|w_* - w_\ell\|^2 + \delta\|w - w_\ell\|^2) \\ &\le \delta^{-1}\|\sigma_y(\lambda_{P,\ell} - \mu_P)\|^2 + \sigma_y\delta\|w - w_\ell\|^2 + \Psi(p_\ell) - (\mu_P, \sigma_y p_\ell)_0 \\ &\quad + \nu_1(\delta^{-1}\|w_* - w_\ell\|^2 + \delta\|w - w_\ell\|^2).\end{aligned}$$

We choose δ with $\delta < \nu_0/(\sigma_y + \nu_1)$. Subtraction of $(\sigma_y + \nu_1)\delta\|w - w_\ell\|$, division by $\nu_0 - (\sigma_y + \nu_1)\delta$ and the estimate (4.6) for $\|w_* - w_\ell\|^2$ complete the proof. □

It remains to show that this estimate is also an upper bound of the combined error of the primal variables and the Lagrange multiplier. To this end, we once again use the variational equality (4.5).

Lemma 4.5. *We have*

$$\|\lambda_P - \lambda_{P,\ell}\|_0 \lesssim \|w - w_\star\|_W.$$

Proof. The assertion follows from

$$\begin{aligned}\sigma_y\|\lambda_P - \lambda_{P,\ell}\|_0 = \sigma_y \sup_{q\in Q,\|q\|_0=1} (\lambda_P - \lambda_{P,\ell}, q)_{0,\Omega} &= \sup_{z\in W,\|z\|_0=1} (\lambda_P - \lambda_{P,\ell}, \sigma_y q)_{0,\Omega} \\ &= \sup_{z\in W,\|z\|=1} a(w_\star - w, z) \le \nu_0\|w - w_\star\|_W.\end{aligned}$$

□

The combination of the estimate (4.6), Lemma 4.5 and Theorem 4.4 directly show the next result.

Corollary 4.6. *Let* w, λ_P *be the solution of* (1.16) *and* $w_\ell, \lambda_{P,\ell}$ *be the solution of* (2.3),(2.3). *The error of the primal solution can be estimated as follows*

$$\begin{aligned}\|w - w_\ell\|^2 + \|\lambda_P - \lambda_{P,\ell}\|_0^2 \lesssim \eta_{P,\ell}^2 &+ \|\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y\lambda_{P,\ell}\|_0^2 \\ &+ \|\lambda_{P,\ell} - \mu_P\|_0^2 + \Psi(p_\ell) - (\mu_P, \sigma_y p_\ell)_0\end{aligned}$$

with arbitrary $\mu_P \in \Lambda_P$.

The estimates still include an unspecified $\mu_P \in \Lambda_P$ which is present due to the possibly non-conform choice of the set of discrete Lagrange multipliers, i.e. $\Lambda_{P,\ell} \nsubseteq \Lambda_P$. The choice of μ_P as

$$\overline{(\lambda_{P,\ell})}_1 := \begin{cases} \lambda_{P,\ell} & \text{for } \lambda_{P,\ell} : \lambda_{P,\ell} \leq 1, \\ \lambda_{P,\ell}/(\lambda_{P,\ell} : \lambda_{P,\ell})^{1/2} & \text{else} \end{cases}$$

seems to be reasonable since it somehow measures the distance of $\lambda_{P,\ell}$ to Λ_P. In practice, the exact numerical integration of the terms involving $\overline{(\lambda_{P,\ell})}_1$ would require the usage of an extra Gaussian quadrature rule in between the points of $\mathcal{G}^d$ corresponding to the polynomial degree of $\lambda_{P,\ell}$. However, the exactness can not be easily guaranteed if the set of points where $\lambda_{P,\ell} : \lambda_{P,\ell}$ changes to values greater one remains unknown.

Once again we emphasize that for the discretization with at most piecewise bilinear or trilinear functions the given discretization of the Lagrange multiplier is conform. Thus, we can choose $\mu_P = \lambda_{P,\ell}$ in Corollary 4.6 and get

$$\|w - w_\ell\|_W^2 + \|\lambda_P - \lambda_{P,\ell}\|_0^2 \lesssim \eta_{P,\ell}^2 + \|\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y \lambda_{P,\ell}\|_0^2. \tag{4.7}$$

If we further assume $\operatorname{dev}(\sigma(w_\ell)) \in Q_\ell$, we can use Lemma 4.3 to conclude the reliability and efficiency of η_ℓ.

Corollary 4.7. *Let the polynomial degree k of Q_ℓ be less or equal to one and let hold $\operatorname{dev}(\sigma(z_\ell)) \in Q_\ell$ for all $z_\ell \in W_\ell$. The estimator $\eta_{P,\ell}$ is reliable, and efficient up to data oscillations in the sense that*

$$\|w - w_\ell\|_W^2 + \|\lambda_P - \lambda_{P,\ell}\|_0^2 \lesssim \eta_{p,\ell}^2 \lesssim \|w - w_\ell\|_W^2 + \|\lambda_P - \lambda_{P,\ell}\|_0^2 + \operatorname{osc}_\ell^2(f_\Omega) + \operatorname{osc}_\ell^2(f_N).$$

Proof. The reliability directly follows from Lemma 4.3 and the estimate (4.7). The efficiency is a direct consequence of Lemma 4.3 and standard arguments applied to the auxiliary variational equality (4.5). ☐

4.1.2 Frictional Contact

In the same way as for the elastoplastic problem without contact conditions, we use ideas of [87, 20] and introduce the elliptic variational equation

$$a(w_*, z) = \mathcal{F}(z) - b(\lambda_{\mathscr{L}}, \gamma(z)) - (\lambda_{P,\ell}, q)_0. \tag{4.8}$$

The solution w_* of (4.8) and its finite element approximation exist and are unique due to standard arguments for linear elliptic problems [22]. Obviously, the first component w_ℓ of the solution to the discrete stationary condition (1.45) and the finite element approximation of w_* coincide. Thus, standard arguments for linear variational equalities [107] show

$$\|w_* - w_\ell\|_W^2 \lesssim \eta^2 := \sum_{T \in \mathcal{T}_\ell} \eta^2(T) \tag{4.9}$$

with

$$\eta^2(T) := |T| R_T^2 + \sum_{E \in \mathcal{E}_T} |T|^{1/2} R_E^2$$

and

$$R_T := \|f_\Omega + \operatorname{div} \sigma(w_\ell)\|_{0,T} + |T|^{-1} \| \operatorname{dev}(\sigma(w_\ell) - \mathbb{H} p_\ell) - \sigma_y \lambda_{P,\ell}\|_{0,T}, \tag{4.10}$$

$$R_E := \begin{cases} \|[\sigma(w_\ell)]\nu_E\|_{O,E}, & E \in \mathcal{E}_\ell^\circ, \\ \|f_N - \sigma(w_\ell)\nu_E\|_{O,E}, & E \in \mathcal{E}_\ell^N, \\ \|\lambda_{\mathscr{L}} + \sigma(w_\ell)\nu_E\|_{O,E}, & E \in \mathcal{E}_\ell^C, \\ 0, & E \subset \mathcal{E}_\ell^D. \end{cases} \tag{4.11}$$

We note again that the term $\| \operatorname{dev}(\sigma(w_\ell) - \mathbb{H} p_\ell) - \sigma_y \lambda_{P,\ell}\|_{0,T}$ equals zero if $\operatorname{dev}(\sigma(w_\ell)) \in Q_h^k$. For example, this is the case whenever the mesh consists only of triangles and we choose $k = l - 1$. Moreover, $\lambda_{P,\ell}$ is the L^2-projection of $\operatorname{dev}(\sigma(w_\ell) - \mathbb{H} p_\ell)/\sigma_y$ onto the space Q_ℓ.

It is easy to see that the error $\|w_* - w_\ell\|_W$ of the auxiliary problem is equivalent to the residual

$$\langle \operatorname{Res}(w_\ell), z \rangle := \mathcal{F}(z) - a(w_\ell, z)_0 - b(\lambda_{\mathscr{L}}, z) - (\lambda_{P,\ell}, \sigma_y q)_0,$$

in the sense that

$$\| \operatorname{Res}(w_\ell)\|_{W'} \lesssim \|w_* - w_\ell\|_W \lesssim \| \operatorname{Res}(w_\ell)\|_{W'}. \tag{4.12}$$

This again follows by standard arguments for variational equalities as in [107]. Note that any other estimator $\tilde{\eta}$ of the residual instead of η would yield the same result.

We aim to use the estimate for the discretization error of the auxiliary problem to bound the error $\|w - w_\ell\|_W$. To this end, we will have to cope with the additional terms. In particular, we have to control the term $\langle \lambda_C, \gamma_n(w_\ell) - g \rangle$. The following lemma gives an estimate in terms of a posteriori known quantities and the discretization error. In [94], this result is shown for the linear elastic Signorini problem. It is easy to see that the arguments still hold for the elastoplastic version.

Lemma 4.8. *It holds*

$$\langle -\lambda_C, g - \gamma_n(w_\ell) \rangle \lesssim \delta \|w - w_\ell\|_W^2 + (4\delta)^{-1} \|(g - \gamma_n(w_\ell))_+\rangle\|_{1/2}^2 + \eta^2 + |\langle \lambda_{C,\mathscr{L}}, (g - \gamma_n(w_\ell))_+|$$

for arbitrary $\delta > 0$.

Proof. We follow the arguments in [94, 87]. Let $z = (v, 0) \in \ker(\gamma_t)$ with $\gamma_n(z) = (g - \gamma_n(w_\ell))_+$. From the equality (1.40), continuity of a and Res, and Young's inequality, we conclude

$$\begin{aligned}\langle -\lambda_C, g-\gamma_n(w_\ell)\rangle &= \langle -\lambda_C, g-\gamma_n(w_\ell)+(g-\gamma_n(w_\ell))_+\rangle + \langle \lambda_C, (g-\gamma_n(w_\ell))_+\rangle\\ &\leq \langle \lambda_C, (g-\gamma_n(w_\ell))_+\rangle\\ &= \mathcal{F}(z) - a(w,z)\\ &= a(w_\ell - w, z) + (\lambda_{C,\mathscr{L}}, \gamma_n(z))_0 + \langle \mathrm{Res}(w_\ell), z\rangle\\ &\lesssim \delta\|w-w_\ell\|_W^2 + (4\delta)^{-1}\|z\|_W^2 + \eta^2 + |(\lambda_{C,\mathscr{L}}, (g-\gamma_n(w_\ell))_+)|.\end{aligned}$$

The trace inequality implies $\|z\|_W \lesssim \|(g-\gamma_n(w_\ell))_+\|_{1/2}$, which completes the proof. □

Remark 4.9. We observe that we could replace the $(g-\gamma_n(w_\ell))_+$ by any other $\tilde{z}$ with $g-\gamma_n(w_\ell)+\tilde{z} \in H_+^{1/2}(\Gamma_C)$. However, we focus on $(g-\gamma_n(w_\ell))_+$ here as it is a good measure of the discretization error in the non-penetration condition.

We are now able to state an a posteriori estimate for the error $\|w-w_\ell\|$.

Theorem 4.10. *Let w, λ, λ_P be the solution of* (1.45) *and $w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}$ be the solution of* (2.13). *The error of the primal solution can be estimated as follows*

$$\|w-w_\ell\|_W^2 \lesssim \bar{\eta}^2(\mu,\mu_P)$$

with

$$\begin{aligned}\bar{\eta}^2(\mu,\mu_P) := \eta^2 &+ \|\lambda_{\mathscr{L}} - \mu\|_{-1/2}^2 + \|\lambda_{P,\ell}-\mu_P\|_0^2 + \Psi_F(w_\ell) - \langle \mu_F, \tilde{f}_F\gamma_t(w_\ell)\rangle\\ &+ \|(g-\gamma_n(w_\ell))_+\|_{1/2}^2 + |\langle \lambda_{C,\mathscr{L}}, (g-\gamma_n(w_\ell))_+\rangle| + |\langle \mu_C, g-\gamma_n(w_\ell)\rangle|\\ &+ \Psi(p_\ell) - (\mu_P, \sigma_y p_\ell)_0,\end{aligned}$$

and for arbitrary $\mu = (\mu_C, \mu_F) \in \Lambda$ and $\mu_P \in \Lambda_P$.

Proof. We have

$$\begin{aligned}\|w-w_\ell\|^2 &\lesssim a(w-w_\ell, w-w_\ell) = a(w-w_*, w-w_\ell) + a(w_*-w_\ell, w-w_\ell)\\ &\leq b(\lambda_{\mathscr{L}}-\lambda, \gamma(w-w_\ell)) + (\lambda_{P,\ell}-\lambda_P, \sigma_y(p-p_\ell))_0 + \nu_1\|w_*-w_\ell\|\|w-w_\ell\|\\ &\leq b(\lambda_{\mathscr{L}}-\mu, \gamma(w-w_\ell)) + b(\mu-\lambda, \gamma(w-w_\ell)) + (\lambda_{P,\ell}-\mu_P, \sigma_y(p-p_\ell))_0\\ &\quad + (\mu_P - \lambda_P, \sigma_y(p-p_\ell))_0 + \nu_1(4\delta)^{-1}\|w_*-w_\ell\|^2 + \nu_1\delta\|w-w_\ell\|^2\\ &\leq \|\lambda_{\mathscr{L}}-\mu\|_{-1/2}\|\gamma(w-w_\ell)\|_{1/2} + \sigma_y\|\lambda_{P,\ell}-\mu\|_0\|w-w_\ell\|_W + b(\mu-\lambda, \gamma(w)-\tilde{g})\\ &\quad + \langle \mu_C - \lambda_C, g-\gamma_n(w_\ell)\rangle + \Psi_F(w_\ell) - \langle \mu_F, \tilde{f}_F\gamma_t(w_\ell)\rangle + \Psi(p_\ell) - (\mu_P, \sigma_y p_\ell)_0\\ &\quad + (4\delta)^{-1}\|w_*-w_\ell\|_W^2 + \delta\|w-w_\ell\|_W^2.\end{aligned}$$

The estimate (4.9) for $\|w_*-w_\ell\|_W^2$ and Lemma 4.8 together with the trace and Young's inequalities yield the assertion. □

It remains to specify how the term $\|(g-\gamma_n(w_\ell))_+\|_{1/2}$ can be evaluated within an actual computation. To this end, we follow the arguments of [94, Remark 5.4]. If the gap function fulfills $g \in H^1(\Gamma_C)$ then the positive part fulfills $(g - \gamma_n(w_\ell))_+ \in H^1(\Gamma_C)$ see [52, Corollary 1.2.1]. In this case, we can employ the results of [105, Chapter 1.3.3] to estimate the $H^{1/2}$ norm by the H^1 and the L^2 norm. In practice, it additionally shows to be sufficient to simply use the L^2 norm to estimate the $H^{-1/2}$ norm.

With the help of an inf-sup condition for both Lagrange multipliers λ_C and λ_F we can show that additionally the discretization error in all Lagrange multipliers is controlled by the full residual error estimator of Theorem 4.10.

Theorem 4.11. *Let the inf-sup condition* (1.26) *hold. Then*

$$\|w - w_\ell\|_W^2 + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2}^2 + \|\lambda_P - \lambda_{P,\mathscr{L}}\|_0^2 \lesssim \bar{\eta}^2(\mu, \mu_P)$$

Proof. From the inf-sup condition (1.26) and the stationary condition of Lemma (1.15), we conclude

$$\begin{aligned}
&\alpha\|\lambda - \lambda_{\mathscr{L}}\|_{-1/2} + \|\lambda_P - \lambda_{P,\ell}\|_0 \\
&\quad\leq \sup_{z\in W, \|z\|=1} b(\lambda - \lambda_{\mathscr{L}}, z) + (\lambda_P - \lambda_{P,\ell}, \sigma_y q)_0 \\
&\quad= \sup_{z\in W, \|z\|=1} \mathcal{F}(z) - a(w, z) - b(\lambda_{\mathscr{L}}, z) - (\lambda_{P,\ell}, \sigma_y q)_0 \\
&\quad\leq \sup_{z\in W, \|z\|=1} \langle \mathrm{Res}(w_\ell), z\rangle + a(w_\ell - w, z) \\
&\quad\leq \|\operatorname{Res}\|_{W'} + \nu_1\|w - w_\ell\|_W.
\end{aligned}$$

The assertion now directly follows from Theorem 4.10 and the estimates (4.9) and (4.12). □

It remains to choose μ and μ_P so that they can be easily computed from the given data and all terms involving them tend to zero as the level ℓ increases. An obvious choice is to use the discrete Lagrange multipliers and cut them off in such a way that they belong to Λ and Λ_P, respectively. However, this again results in functions which are no longer integrated exactly by the used Gauss quadrature rules. We set

$$(f)_+(x) := \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{else} \end{cases}$$

and

$$\overline{(f)}_1(x) := \begin{cases} f(x) & \text{if } |f(x)| \leq 1 \\ f(x)/|f(x)| & \text{else.} \end{cases}$$

Corollary 4.12. *The error satisfies*

$$\|w - w_\ell\|_W^2 + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2}^2 + \|\lambda_P - \lambda_{P,\mathscr{L}}\|_0^2 \lesssim \bar{\eta}^2((\lambda_{C,\mathscr{L}})_+, \overline{(\lambda_{F,\mathscr{L}})}_1, \overline{(\lambda_{P,\ell})}_1).$$

Proof. The assertion follows directly from the previous theorem as well as $\overline{(\lambda_{P,\ell})}_1 \in \Lambda_P$ and $((\lambda_{C,\mathscr{L}})_+, \overline{(\lambda_{F,\mathscr{L}})}_1) \in \Lambda$. $\square$

For the case of $\Lambda_{\mathscr{L}} \subset \Lambda$ and $\Lambda_{P,\ell} \subset \Lambda_P$ which holds for at most bilinear or trilinear basis functions, we have $\overline{(\lambda_{P,\ell})}_1 = \lambda_{P,\ell}$ and $((\lambda_{C,\mathscr{L}})_+, \overline{(\lambda_{F,\mathscr{L}})}_1) = (\lambda_{F,\mathscr{L}}, \lambda_{C,\mathscr{L}})$. Thus, the error estimator reduces to

$$\hat{\eta}_\ell^2 := \eta_\ell^2 + |\langle \lambda_{C,\mathscr{L}}, g - \gamma_n(w_\ell)\rangle| + \Psi_F(w_\ell) - (\mu_F, \tilde{f}_F \gamma_t(w_\ell))_0 + \Psi(p_\ell) - (\mu_P, \sigma_y p_\ell)_0.$$

Corollary 4.13. *Let the polynomial degrees of the finite element spaces be $l, k, m \leq 1$. Then*

$$\|w - w_\ell\|_W^2 + \|\lambda - \lambda_{\mathscr{L}}\|_{-1/2}^2 + \|\lambda_P - \lambda_{P,\mathscr{L}}\|_0^2 \lesssim \hat{\eta}_\ell^2.$$

Proof. We have

$$\begin{aligned}
\|w - w_\ell\|_W^2 &\lesssim a(w - w_\ell, w - w_\ell) = a(w - w_*, w - w_\ell) + a(w_* - w_\ell, w - w_\ell) \\
&\lesssim b(\lambda_{\mathscr{L}} - \lambda, \gamma(w - w_\ell)) + (\lambda_{P,\ell} - \lambda_P, \sigma_y(p - p_\ell))_0 + \|w_* - w_\ell\|_W \|w - w_\ell\|_W \\
&\lesssim \langle \lambda_{C,\mathscr{L}} - \lambda_C, \gamma_n(w) - g\rangle + \langle \lambda_{C,\mathscr{L}} - \lambda_C, g - \gamma_n(w_\ell)\rangle \\
&\qquad + \Psi_F(w_\ell) - (\lambda_{F,\mathscr{L}}, \tilde{f}_F \gamma_t(w_\ell))_0 + \Psi(p_\ell) - (\lambda_{P,\ell}, \sigma_y p_\ell)_0 \\
&\qquad + (4\delta)^{-1} \|w_* - w_\ell\|_W^2 + \delta \|w - w_\ell\|_W^2.
\end{aligned}$$

Since the polynomial degrees are smaller than one $\Lambda_{C,\mathscr{L}} \subset \Lambda_C$ and therefore $\langle -\lambda_C, g - \gamma_n(w_\ell)\rangle \leq 0$. This, the stationary condition (2.13), and the estimate (4.9) on $\|w_* - w_\ell\|_W^2$ yield the assertion. $\square$

4.2 Dual weighted residual estimator

In the application of the finite element method to engineering problems, it is often useful to control the discretization error of a functional $J : W \times \Lambda \times \Lambda_P \to \mathbb{R}$ which measures a specific quantity of interest, see for example [15]. In contrast, the residual estimates of the previous sections only controls the discretization error with respect to the norms of the underlying spaces. The approaches which seek to control such functionals are known as goal oriented. They can be based on the energy norm but we focus on dual weighted residual error estimation (DWR) proposed by Becker and Rannacher [12] for an overview on DWR see the monograph [10]. The DWR method has been applied to the Signorini problem in [17]. Recently, an estimate was derived for the mixed formulation of the Signorini problem in linear elasticity [96] which allows the functional to depend on the Lagrange multipliers. For a regularized problem in elastoplasticity a DWR error estimator can be found in [81].

In the DWR method, the error in the quantity of interest is estimated with the help of the solution of a dual problem or more precisely by the residual and the dual residual which we define below. The transfer of the arguments in [80, 96] to the frictional contact problem with elastoplastic material behavior of Section 1.5 is straightforward. We start by expressing $J(u, \lambda, \lambda_P) - J(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell})$ in terms of the dual weighted residuals. This done, we suggest an approximation for the analytical solutions included in the dual weighted residuals in order to eventually estimate $J(u, \lambda, \lambda_P) - J(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell})$.

We consider $J(z, \mu, \mu_p)$ to be three times Fréchet differentiable and denote the derivatives with respect to z,μ and μ_p by J'_w, J'_λ and J'_{λ_P}, respectively. Similair to [80], the dual problem is then given by the following mixed formulation.

Definition 4.14. *The dual problem consists in finding* $(\zeta, \xi, \xi_P) \in W \times M \times M_P$ *such that*

$$\begin{aligned} a(z, \zeta) + b(\xi, z) + (\xi_P, \sigma_y q)_0 &= \langle J'_w(w, \lambda, \lambda_P), z\rangle \\ b(\mu, \zeta) &= \langle J'_\lambda(w, \lambda, \lambda_P), \mu\rangle \\ (\mu_P, \sigma_y \zeta_P)_0 &= \langle J'_P(w, \lambda, \lambda_P), \mu_P\rangle \end{aligned}$$

holds for all $(z, \mu, \mu_P) \in W \times M \times M_P$.

The existence and uniqueness of the dual solution is guaranteed by the same arguments which we used for the saddle point problems of Chapter 1.

Definition 4.15. *The approximation of the dual solution by finite elements is given as to find* $(\zeta_\ell, \xi_{\mathscr{L}}, \xi_{P,\mathscr{L}}) \in W_\ell \times M_{\mathscr{L}} \times M_{P,\mathscr{L}}$ *such that*

$$\begin{aligned} a(z_\ell, \zeta_\ell) + b(\xi_{\mathscr{L}}, z_\ell) + (\xi_{P,\mathscr{L}}, \sigma_y q_\ell)_0 &= \langle J'_w(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), z_\ell\rangle \\ b(\mu_{\mathscr{L}}, \zeta_\ell) &= \langle J'_\lambda(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), \mu_{\mathscr{L}}\rangle \\ (\mu_{P,\ell}, \sigma_y \zeta_{P,\ell})_0 &= \langle J'_P(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), \mu_{P,\ell}\rangle \end{aligned} \tag{4.13}$$

holds for all $(z_\ell, \mu_{\mathscr{L}}, \mu_{P,\ell}) \in W_\ell \times M_{\mathscr{L}} \times M_{P,\ell}$.

We recall that the residual $\mathrm{Res}(z) \in W'$ is defined by

$$\langle \mathrm{Res}(w), z\rangle := \mathcal{F}(z) - b(\lambda_{\mathscr{L}}, z) - (\lambda_{P,\ell}, \sigma_y q)_0 - a(w, z).$$

Next, we introduce the dual residuals $\mathrm{Res}^*(z) \in W'$, $\mathrm{Res}^*_\lambda(z) \in M'$ and $\mathrm{Res}^*_P(z) \in M'_P$ as

$$\begin{aligned} \langle \mathrm{Res}^*(w), z\rangle &:= \langle J'_w(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), z\rangle - b(\xi_{\mathscr{L}}, z) - (\xi_{P,\ell}, \sigma_y q)_{0,\Omega} - a(z, w), \\ \langle \mathrm{Res}^*_\lambda(w), \mu\rangle &:= \langle J'_\lambda(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), \mu\rangle - b(\mu, w), \\ \langle \mathrm{Res}^*_P(w), \mu_P\rangle &:= \langle J'_P(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), \mu_P\rangle - (\mu_P, \sigma_y p)_{0,\Omega}. \end{aligned}$$

In order to express the error $J(w, \lambda, \lambda_P) - J(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell})$ in terms of the residuals we make use of the following equalities. By definition of the dual residuals and the discrete dual problem, we directly conclude

$$\langle \mathrm{Res}^*(\zeta_\ell), z_\ell\rangle = \langle \mathrm{Res}^*_\lambda(\zeta_\ell), \mu_{\mathscr{L}}\rangle = \langle \mathrm{Res}^*_P(\zeta_\ell), \mu_{P,\ell}\rangle = 0 \tag{4.14}$$

for the solution ζ_ℓ of (4.13) and all $(z_\ell, \mu_{\mathscr{L}}, \mu_{P,\ell}) \in W_\ell \times M_{\mathscr{L}} \times M_{P,\ell}$. Moreover, the discrete solution w_ℓ fulfills

$$\langle \mathrm{Res}(w_\ell), z_\ell \rangle = 0$$

for all $z_\ell \in W_\ell$ and this further implies

$$\begin{aligned} a(w - w_\ell, z) &= \mathcal{F}(z) - b(\lambda, z) - (\lambda_P, \sigma_y q)_0 - a(w_\ell, z) \\ &= \mathcal{F}(z) - b(\lambda_{\mathscr{L}}, z) - (\lambda_{P,\ell}, \sigma_y q)_0 - a(w_\ell, z) \\ &\quad + b(\lambda_{\mathscr{L}} - \lambda, z) + (\lambda_{P,\ell} - \lambda_P, \sigma_y q)_0 \\ &= \langle \mathrm{Res}(w_\ell), z - z_\ell \rangle + b(\lambda_{\mathscr{L}} - \lambda, z) + (\lambda_{P,\ell} - \lambda_P, \sigma_y q)_0. \end{aligned} \tag{4.15}$$

For $\kappa \in (0, 1)$, we set

$$w_\kappa := w - \kappa w_\ell, \quad \lambda_\kappa := \lambda - \kappa \lambda_{\mathscr{L}}, \quad \lambda_{P,\kappa} := \lambda_P - \kappa \lambda_{P,\ell},$$

and use the fundamental theorem of calculus to conclude

$$\begin{aligned} &J(w, \lambda, \lambda_P) - J(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}) \\ &= \int_0^1 \langle J'_w(w_\kappa, \lambda_\kappa, \lambda_{P,\kappa}), w - w_\ell \rangle + \langle J'_\lambda(w_\kappa, \lambda_\kappa, \lambda_{P,\kappa}), \lambda - \lambda_{\mathscr{L}} \rangle \\ &\quad + \langle J'_P(w_\kappa, \lambda_\kappa, \lambda_{P,\kappa}), \lambda - \lambda_{P,\ell} \rangle \, d\kappa \\ &= \frac{1}{2} \big(\langle J'_w(w, \lambda, \lambda_P), w - w_\ell \rangle + \langle J'_\lambda(w, \lambda, \lambda_P), \lambda - \lambda_{\mathscr{L}} \rangle \\ &\quad + \langle J'_P(w, \lambda, \lambda_P), \lambda - \lambda_{P,\ell} \rangle \\ &\quad + \langle J'_w(w_\ell, \lambda_\ell, \lambda_{P,\ell}), w - w_\ell \rangle + \langle J'_\lambda(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), \lambda - \lambda_{\mathscr{L}} \rangle \\ &\quad + \langle J'_P(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}), \lambda - \lambda_{P,\ell} \rangle \big) \\ &\quad + R(w - w_\ell, \lambda - \lambda_{\mathscr{L}}, \lambda_P - \lambda_{P,\ell}) \end{aligned} \tag{4.16}$$

with

$$\begin{aligned} &R(w - w_\ell, \lambda - \lambda_{\mathscr{L}}, \lambda_P - \lambda_{P,\ell}) \\ &\qquad := \frac{1}{2} \int_0^1 \langle J'''(w_\kappa, \lambda_\kappa, \lambda_{P,\kappa})[(z, \mu, \mu_P)][(z, \mu, \mu_P)], (z, \mu, \mu_P) \rangle \kappa(\kappa - 1) \, d\kappa \end{aligned}$$

being the remainder of the trapezoidal rule. Now, we are able to proof the following theorem.

Theorem 4.16. *There holds*

$$\begin{aligned}
&J(w,\lambda,\lambda_P) - J(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell})\\
&= \frac{1}{2}\Big(\langle \mathrm{Res}(w_\ell), \zeta - \tilde{z}_\ell\rangle\\
&\quad + \langle \mathrm{Res}^*(\zeta_\ell), w - z_\ell\rangle + \langle \mathrm{Res}^*_\lambda(\zeta_\ell), \lambda - \mu_{\mathscr{L}}\rangle + \langle \mathrm{Res}^*_P(\zeta_\ell), \lambda_P - \mu_{P,\ell}\rangle\\
&\quad + b(\xi + \xi_{\mathscr{L}}, w - w_\ell) + (\xi_P + \xi_{P,\ell}, \sigma_y(q - q_\ell))_0\Big)\\
&\quad + R(w - w_\ell, \lambda - \lambda_{\mathscr{L}}, \lambda_P - \lambda_{P,\ell})
\end{aligned}$$

for all $\tilde{z}_\ell$, $z_\ell \in W_\ell$, $\mu_{\mathscr{L}} \in M_{\mathscr{L}}$ *and* $\mu_{P,\ell} \in M_\ell$.

Proof. From the definition of the dual problem, the equality (4.15) and the definition of the residuals, we have

$$\begin{aligned}
&\langle J'_w(w,\lambda,\lambda_P) + J'_w(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), w - w_\ell\rangle + \langle J'_\lambda(w,\lambda,\lambda_P) + J'_\lambda(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), \lambda - \lambda_{\mathscr{L}}\rangle\\
&\quad + \langle J'_P(w,\lambda,\lambda_P) + J'_P(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), \lambda_P - \lambda_{P,\ell}\rangle\\
&= a(w - w_\ell, \zeta) + b(\xi, w - w_\ell) + (\xi_P, \sigma_y(p - p_\ell))_0 + b(\lambda - \lambda_{\mathscr{L}}, \zeta) + (\lambda_P - \lambda_{P,\ell}, \zeta_P)_0\\
&\quad + \langle J'_w(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), w - w_\ell\rangle + \langle J'_\lambda(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), \lambda - \lambda_{\mathscr{L}}\rangle\\
&\quad + \langle J'_P(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), \lambda_P - \lambda_{P,\ell}\rangle\\
&= \langle \mathrm{Res}(w_\ell), \zeta - \tilde{z}_\ell\rangle + \langle J'_w(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), w - w_\ell\rangle - b(\xi_{\mathscr{L}}, w - w_\ell) - (\xi_{P,\mathscr{L}}, w - w_\ell)_0\\
&\quad - a(w - w_\ell, \zeta_\ell) + \langle J'_\lambda(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), \lambda - \lambda_{\mathscr{L}}\rangle - b(\lambda - \lambda_{\mathscr{L}}, \zeta)\\
&\quad + \langle J'_P(w_\ell,\lambda_{\mathscr{L}},\lambda_{P,\ell}), \lambda_P - \lambda_{P,\ell}\rangle - (\lambda_P - \lambda_{P,\ell}, \sigma_y(p - p_\ell))_0\\
&\quad + b(\xi + \xi_{\mathscr{L}}, w - w_\ell) + (\xi_P + \xi_{P,\ell}, \sigma_y(p - p_\ell))_0\\
&= \langle \mathrm{Res}(w_\ell), \zeta - \tilde{z}_\ell\rangle + \langle \mathrm{Res}^*(\zeta_\ell), w - w_\ell\rangle + \langle \mathrm{Res}^*_w(\zeta_\ell), \lambda - \lambda_{\mathscr{L}}\rangle\\
&\quad + \langle \mathrm{Res}(\zeta_\ell), \lambda_P - \lambda_{P,H}\rangle + b(\xi + \xi_{\mathscr{L}}, w - w_\ell) + (\xi_P + \xi_{P,\ell}, \sigma_y(p - p_\ell))_0\\
&= \langle \mathrm{Res}(w_\ell), \zeta - \tilde{z}_\ell\rangle + \langle \mathrm{Res}^*(\zeta_\ell), w - z_\ell\rangle + \langle \mathrm{Res}^*_w(\zeta_\ell), \lambda - \mu_{\mathscr{L}}\rangle\\
&\quad + \langle \mathrm{Res}(\zeta_\ell), \lambda_P - \mu_{P,H}\rangle + b(\xi + \xi_{\mathscr{L}}, w - w_\ell) + (\xi_P + \xi_{P,\ell}, \sigma_y(p - p_\ell))_0
\end{aligned}$$

where in the last equation we used the identity (4.14). Plugged into the equation (4.16) this yields the assertion. □

As mentioned before, the name dual weighted residuals comes from the fact that the residuals are weighted by the error of the respective dual problem as seen in the previous theorem. This theorem is basic identity of the DWR method for elastoplasticity with linear kinematic hardening and frictional contact conditions. However, it does not really imply a practicable scheme since the right hand side of the identity in Theorem 4.16 still includes the unknown analytic solutions of the frictional contact problem for elastoplasticity 1.45 and the dual problem 4.14. At this point, the derivation of the DWR error estimator leaves the strict analysis and we turn to a more heuristic approach. We follow the common approaches found in [10] and known to give good results in applications. Thus, we replace the solution w, the Lagrange multiplier λ, the dual solution ζ and the dual multiplier ξ by approximations in order to arrive at an computable estimate. Moreover, we omit the remainder R. This can be justified by the fact that R is of higher order.

From here to the end of this section, we restrict the discretization schemes to basis functions of degree one for V_ℓ, i.e. $V_\ell = V_h^1$ the space of piecewise affine, bilinear or trilinear functions on triangles and tetrahedrons, quadrilaterals or hexahedrons, respectively. Moreover, we only consider $Q_\ell = Q_h^0$ and $M_{\mathscr{L}} = M_H^0$, i.e. piecewise constant functions.

We want to approximate the solutions w and ζ by quadratic interpolation on coarser quadrilaterals containing one nodal patch. To guarantee such a patch structure, we follow the ideas in [96]. Hence, we additionally mark all siblings of a cell marked by the estimator, i.e., all cells which were generated by the previous refinement of a coarse cell. This structure given, we use the usual quadratic interpolant I_{2h}^2 based on the nodal values of the patch included in each coarse element. The averaging operators $A : M_{\mathscr{L}} \to V_H^1(\Gamma_C)$ and $A_P : Q_\ell \to Q_h^1 \cap H^1(\Omega, \mathbb{R}^{d\times d}_{\text{sym}})$ are defined as

$$A(\mu_{\mathscr{L}})(x) := \sum_{\xi \in \mathcal{N}_{\mathscr{L}}(\Gamma_C)} \Big(\sum_{E \in \omega(\xi)} |E|\Big)^{-1} \sum_{E \in \omega(\xi)} \mu_{\mathscr{L}|E}(\xi)\varphi_\xi(x), \tag{4.17}$$

$$A_P(q_\ell)(x) := \sum_{\xi \in \mathcal{N}_\ell(\Omega)} \Big(\sum_{K \in \omega(\xi)} |K|\Big)^{-1} \sum_{K \in \omega(\xi)} q_{\ell|K}(\xi)\varphi_{P,\xi}(x), \tag{4.18}$$

where $\omega(\xi) := \bigcup_{K, \xi \in \bar{K}} K$ is the usual nodal patch, and φ_ξ and $\varphi_{P,\xi}$ denote the usual nodal basis of the spaces $V_H^1(\Gamma_C)$ and $Q_h^1 \cap H^1(\Omega, \mathbb{R}^{d\times d}_{\text{sym}})$, respectively. For meshes of triangles, we refer to [108] and for how to the apply the results to the DWR method to [32].

With the notation $\mathcal{I}(w) := (I_{2h}^2(u), A_P(p))$, we have

$$\begin{aligned}
&J(w, \lambda, \lambda_P) - J(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}) \\
&\approx \frac{1}{2}\Big(\langle \operatorname{Res}(w_\ell), \mathcal{I}(\zeta_\ell) - \zeta_\ell\rangle + \langle \operatorname{Res}^*(\zeta_\ell), \mathcal{I}(w_\ell) - w_\ell\rangle + \langle \operatorname{Res}^*_\lambda(\zeta_\ell), A(\lambda_{\mathscr{L}}) - \lambda_{\mathscr{L}}\rangle \\
&\quad + \langle \operatorname{Res}^*_P(\zeta_\ell), A_P(\lambda_{P,\ell}) - \lambda_{P,\ell}\rangle + b(A(\xi_{\mathscr{L}}) + \xi_{\mathscr{L}}, \mathcal{I}(w_\ell) - w_\ell) \\
&\quad + (A_P(\xi_{P,\ell}) + \xi_{P,\ell}, \sigma_y(A_P(p_\ell) - p_\ell))_0 \Big).
\end{aligned} \tag{4.19}$$

Here, the symbol $\approx$ denotes *approximately the same* in the classical sense and not *equivalent to* as in the previous sections.

Let $j, j_N : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$, $j_\lambda : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^d \to \mathbb{R}$ and $j_P : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}$ be nonlinear functions which are partially differentiable. We assume that the error functional is given as

$$J(w,\lambda,\lambda_P) := \int_\Omega j(x,w(x))\,dx + \int_{\Gamma_N} j_N(x,w(x))\,dx + \int_{\Gamma_C} j_\lambda(x,w(x),\lambda(x))\,dx + \int_\Omega j_P(x,w(x),\lambda_P(x))\,dx.$$

Integration by parts in (4.19) leads to a localization of the dual weighted residuals on the right-hand side of the approximation. Thus, we are able to use them as local error indicator within an adaptive finite element scheme. The localization results in the representation of the residuals as sums of local residuals over cells and edges, or faces, respectively. The local residuals read as follows

$$\begin{aligned}
\langle R_T(w), z\rangle &:= (f_\Omega + \operatorname{div}\sigma(w), v)_{0,T} - (\sigma(w) + \mathbb{H}p - \lambda_{P,\ell}, q)_{0,T}\\
\langle R_E(w), z\rangle &:= 1/2 \begin{cases} ([\sigma_n(w)], v)_{0,E} & \text{for } E \in \mathcal{E}_\ell^\circ \\ (f_N - \sigma_n(w_\ell), v)_{0,E} & \text{for } E \in \mathcal{E}_\ell^N \\ (-\lambda_{\mathscr{L}} - \sigma_n(w_\ell), v)_{0,E} & \text{for } E \in \mathcal{E}_\ell^C \\ 0 & \text{for } E \in \mathcal{E}_\ell^D \end{cases}\\
\langle R_T^*(\zeta), z\rangle &:= (j'(w_\ell) + \operatorname{div}\sigma(\zeta), v)_{0,T} - (\sigma(\zeta) + \mathbb{H}\zeta_P - \xi_{P,\ell}, q)_{0,T}\\
\langle R_E^*(\zeta), z\rangle &:= 1/2 \begin{cases} ([\sigma_n(\zeta)], v)_{0,E} & \text{for } E \in \mathcal{E}_\ell^\circ \\ (j_N'(w_\ell) - \sigma_n(\zeta_\ell), v)_{0,E} & \text{for } E \in \mathcal{E}_\ell^N \\ (j_{\lambda,w}'(w_\ell, \lambda_{\mathscr{L}}) - \xi_{\mathscr{L}} - \sigma_n(\zeta_\ell), v)_{0,E} & \text{for } E \in \mathcal{E}_\ell^C \\ 0 & \text{for } E \in \mathcal{E}_\ell^D \end{cases}.
\end{aligned}$$

The remaining terms $b(A(\xi_{\mathscr{L}})+\xi_{\mathscr{L}}, \mathcal{I}(w_\ell)-w_\ell)$ and $(A_P(\xi_{P,\ell})+\xi_{P,\ell}, \sigma_y(A_P(p_\ell)-p_\ell))_0$ can be localized without integration by parts. Moreover, we set

$$b_E(\lambda_{\mathscr{L}}, \zeta_\ell) := (\lambda_{C,\mathscr{L}}, \gamma_n(\zeta_\ell))_{0,E} + (\lambda_{F,\mathscr{L}}, \gamma_t(\zeta_\ell))_{0,E}.$$

Together the error contributions are

$$\begin{aligned}
\eta(T) &:= \langle R_T(w_\ell), \mathcal{I}(\zeta_\ell) - \zeta_\ell\rangle + \sum_{E\in\mathcal{E}(T)} \langle R_E(w_\ell), \mathcal{I}(\zeta_\ell) - \zeta_\ell\rangle\\
\eta^*(T) &:= \langle R_T^*(\zeta_\ell), \mathcal{I}(w_\ell) - w_\ell\rangle + \sum_{E\in\mathcal{E}(T)} \langle R_E^*(\zeta), \mathcal{I}(w_\ell) - w_\ell\rangle\\
\eta_\lambda^*(T) &:= \sum_{E\in\mathcal{E}(T)} (j_\lambda'(w_\ell, \lambda_{\mathscr{L}}), A(\lambda_{\mathscr{L}}) - \lambda_{\mathscr{L}})_{0,E} - b_E(A(\lambda_{\mathscr{L}}) - \lambda_{\mathscr{L}}, \zeta_\ell)\\
\eta_P^*(T) &:= (j_P'(w_\ell, \lambda_{P,\ell}), A(\lambda_{P,\ell}) - \lambda_{P,\ell})_{0,T} - (A(\lambda_{P,\ell}) - \lambda_{P,\ell}, \zeta_\ell)_{0,T}\\
\beta^*(T) &:= (A_P(\xi_{P,\ell}) + \xi_{P,\ell}, \sigma_y(A_P(p_\ell) - p_\ell))_{0,T} + \sum_{E\in\mathcal{E}(T)} b_E(A(\xi_{\mathscr{L}}) + \xi_{\mathscr{L}}, \mathcal{I}(w_\ell) - w_\ell).
\end{aligned}$$

Finally, the approximate representation of the error by the localized quantities is given as

$$J(w, \lambda, \lambda_P) - J(w_\ell, \lambda_{\mathscr{L}}, \lambda_{P,\ell}) \approx \frac{1}{2} \sum_{T \in \mathcal{T}_\ell} \eta(T) + \eta^*(T) + \eta^*_\lambda(T) + \eta^*_P(T) + \beta^*(T).$$

Remark 4.17. As in Section 4.1, we can replace the Lagrange multiplier $\lambda_{P,\ell}$ by the L^2-projection of $\operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell)/\sigma_y$ onto Q_ℓ. Hence, we can use the results of this section for a direct low-order finite element discretization without a Lagrange multiplier associated to the plastic dissipation functional.

5 Adaptive finite element method (AFEM)

Typically, adaptive finite element methods (AFEMs) use error estimators to generate a sequence of meshes and, therewith, a sequence of approximations via the four steps

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}. \tag{5.1}$$

In practice, the sequence (5.1) is repeated in a loop until a termination criterion is fulfilled. The input data consists of the functions f_Ω and f_N, the initial mesh $\mathcal{T}_0$ as well as a bulk parameter $0 < \theta \leq 1$. The output is a sequence of meshes $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ and a sequence of discrete solutions.

In the following, we focus on the AFEM for the elastoplastic problem without contact conditions 1.14 and two spatial dimensions. Hence, the sequence (5.1) computes the discrete solutions $(w_\ell)_{\ell \in \mathbb{N}_0}$ in the nested spaces $(W_\ell)_{\ell \in \mathbb{N}_0}$. Throughout this chapter, we assume $W_\ell = V_h^1 \times Q_h^0$ if the mesh consists of only triangles and $W_\ell = V_h^1 \times \hat{Q}_h^1$ for meshes of quadrilaterals. We recall that in both cases we have $\Lambda_{P,\ell} \subset \Lambda_P$ and $\operatorname{dev}(\sigma_\ell) \in Q_\ell$. Moreover, we can only proof the optimal convergence of the stresses for the discretization with the spaces V_h^1 and Q_h^0. For such discretization based on the meshes $\mathcal{T}_\ell$, we give the loop over the sequence (5.1) in pseudo code and some more details.

Input Right hand side functions f_Ω, f_N, initial regular triangulation $\mathcal{T}_0$ of the domain Ω such that one refinement edge is selected for each $T \in \mathcal{T}$ and a marking parameter $0 < \theta \leq 1$.

Loop For $\ell = 0, 1, \ldots$ (until termination) do

`SOLVE` Compute the solution $w_\ell \in W_\ell$ of (2.1).

`ESTIMATE` For all $T \in \mathcal{T}_\ell$, compute the estimated error $\eta_{P,\ell}(T)$.

`MARK` Determine the set $\mathcal{M}_\ell$ of all elements marked for refinement by Dörfler marking/ bulk chasing. Given θ, compute $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ of minimal cardinality $|\mathcal{M}_\ell|$ with

$$\theta \eta_{P,\ell}^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell^2(T) := \eta_\ell^2(\mathcal{M}_\ell). \tag{5.2}$$

and add further elements to $\mathcal{M}_\ell$ using a closure algorithm to avoid hanging nodes.

`REFINE` Refine all elements in $\mathcal{M}_\ell$ with newest vertex bisection, i.e. by the green, blue and bisec3 rules, respectively, see Figure 5.1.

Output Sequences of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ and solutions $(w_\ell)_{\ell\in\mathbb{N}_0}$ in nested spaces $(W_\ell)_{\ell\in\mathbb{N}_0}$.

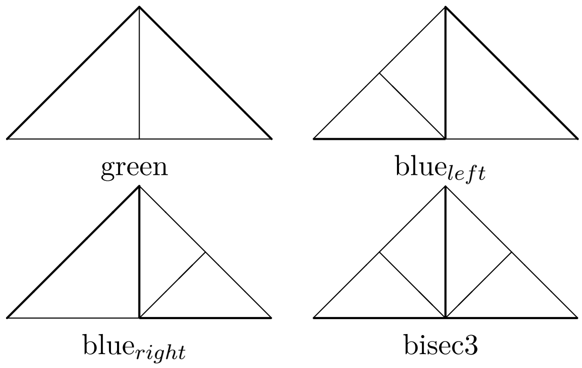


Figure 5.1: Possible refinements of a triangle in the AFEM algorithm. Thick lines denote refinement edges for subsequent newest vertex refinements.

If the mesh is based on quadrilaterals we have to adjust the refinement step within the AFEM algorithm but the other steps remain the same. For meshes of quadrilaterals, the marked cells are refined by quartering. In the closure algorithm further elements are refined such that every edge contains at most an a priori fixed number of hanging nodes. In the case of triangles no hanging nodes are allowed. The estimates on the interpolant in the derivation of the error estimator do not depend on the current level ℓ, if such a closure step is used in order to limit the number of hanging nodes per edge [34]. This makes it possible to show convergence of AFEM based on quadrilaterals, for elliptic variational equations, if Galerkin orthogonality holds for the discrete solution. However, the optimality still remains an open question since it is unclear whether the number of elements refined by the closure algorithm remains bounded.

For the elastoplastic problem without contact conditions, we do not have a variational equality but an inequality and hence Galerkin orthogonality does not hold for the error. For the discretization approach with bilinear/affine functions, we can show the convergence based on the equivalence of the error of the stresses to the error of energies. This equivalence has already been implicitly shown for the affine/constant approach based on triangles in [40].

In the next section, we will give a different proof which also holds for meshes of quadrilaterals. Therefore, we again make use of the stationary condition (2.3),(2.4). We recall that the stationary condition and the variational inequality are equivalent for the affine/constant and the bilinear/affine discretization. Moreover, we use the fact that the discretization of the Lagrange multiplier is conform and fulfills $\lambda_{P,\ell} = (\sigma_\ell - \mathbb{H}p_\ell)/\sigma_y$. To begin with, the following result holds independently of the discretization scheme.

Lemma 5.1. *For the minimizer $w = (u, p) \in W$ of J, there holds*

$$\|w - z\|_W^2 \lesssim E_P(z) - E_P(w). \tag{5.3}$$

for all $z = (v, q) \in W$.

Proof. From the variational inequality (1.14), and the W-ellipticity of a, we conclude

$$\begin{aligned}
\kappa\|z - w\|_W &\leq \frac{1}{2}a(z - w, z - w) \\
&\leq \frac{1}{2}(a(z, z) - a(w, w)) - b(z - w) + \psi(z) - \psi(w) \\
&= E_P(z) - E_P(w). \qquad \square
\end{aligned}$$

The proof of optimal convergence of AFEM for elastoplasticity will also appear in an article [41].

5.1 Adaptive finite element discretization

We set $\sigma := \sigma(u, p)$ and $\sigma_\ell := \sigma(u_\ell, p_\ell)$ with $u_\ell \in V_\ell$ and $p_\ell \in Q_\ell$. The following theorem states that the error of discrete stresses in this norm is equivalent to the error of energies. The result is crucial to show the convergence of AFEM. It is usually shown by the use of Jensen's inequality. In contrast to the similar results of [40], the proof below does not rely on the application of Jensen's inequality and also holds for the bilinear/affine approach based on meshes of quadrilaterals. This is a new result and was the only thing missing in order to show the convergence for a discretization based on quadrilaterals.

Theorem 5.2. *The exact and discrete solutions w and w_ℓ with stress fields σ and σ_ℓ satisfy*

$$\|\sigma - \sigma_\ell\|_0^2 \approx E_P(w_\ell) - E_P(w) \leq (\sigma_\ell - \sigma, \varepsilon(u_\ell - u))_0.$$

Proof. The definition of E_P implies

$$E_P(w_\ell) - E_P(w) = \frac{1}{2}a(w_\ell + w, w_\ell - w) - \mathcal{F}(w_\ell - w) + \psi(w_\ell) - \psi(w). \tag{5.4}$$

The variational inequality (1.14) implies

$$\|\sigma - \sigma_\ell\|_0^2 \lesssim a(w_\ell - w, w_\ell - w) \leq E_P(w_\ell) - E_P(w). \tag{5.5}$$

We note that $(\lambda_{P,\ell}, \sigma_y p_\ell)_0 = \psi(w_\ell)$, $(\lambda_P, \sigma_y p)_0 = \psi(w)$, cf. [55], and $\lambda_{P,\ell} = (\sigma_\ell - \mathbb{H}p_\ell)/\sigma_y$. Moreover, we recall that $(\lambda_{P,\ell} - \lambda, \sigma_y p)_0 \leq 0$ which holds due to the stationary condition (1.17) and $\lambda_{P,\ell} \in \Lambda_P$. We add $a(w_\ell - w, w_\ell - w)/2 \geq 0$ to (5.4) and obtain

$$\begin{aligned}E_P(w_\ell) - E_P(w) &\le a(w_\ell, w_\ell - w) - \mathcal{F}(w_\ell - w) + (\lambda_{P,\ell}, \sigma_y p_\ell)_0 - (\lambda_P, \sigma_y p)_0 \\ &= a(w_\ell, w_\ell - w) - \mathcal{F}(w_\ell - w) + (\lambda_{P,\ell}, \sigma_y(p_\ell - p))_0 + (\lambda_{P,\ell} - \lambda_P, \sigma_y p)_0 \\ &\le a(w_\ell, w_\ell - w) - \mathcal{F}(w_\ell - w) + (\sigma_\ell - \mathbb{H}p_\ell, (p_\ell - p))_0 \\ &= (\sigma_\ell, \varepsilon(u_\ell) - \varepsilon(u))_0 - \mathcal{F}(z_\ell - w) \\ &= (\sigma_\ell - \sigma, \varepsilon(u_\ell) - \varepsilon(u))_0.\end{aligned}$$

To proof the last inequality we use Young's inequality and Lemma 5.1 to conclude

$$\begin{aligned}(\sigma_\ell - \sigma, \varepsilon(u_\ell) - \varepsilon(u)) &\le \frac{\kappa}{2}\|\sigma - \sigma_\ell\|_0^2 + \frac{1}{2\kappa}\|w_\ell - w\|_W^2 \\ &\lesssim \frac{\kappa}{2}\|\sigma - \sigma_\ell\|_0^2 + \frac{1}{2}E_P(w_\ell) - E_P(w).\end{aligned}$$

The equivalence now follows by the subtraction of $(E_P(w_\ell) - E_P(w))/2$, multiplication by 2 and the inequality (5.5). □

The assertion of Theorem 5.2 remains valid for discretization spaces defined on refinements $\mathcal{T}_k$ of $\mathcal{T}_\ell$. We recall that we employ only newest vertex bisections in order to refine meshes of triangles. We call a triangulation admissible if it is a refinement of the shape regular initial triangulation $\mathcal{T}_0$. The set of admissible triangulations is denoted by $\mathbb{T}$. We will use properties of the newest vertex bisection later and the closure algorithm, in a standard way, to show optimal convergence. In particular, we will use the property that two admissible triangulations always have a unique smallest common refinement [42]. We call this common refinement the overlay. For more details on the representation of refinements and meshes via so-called forests and trees, we refer to [42].

Corollary 5.3. *For $m \ge 1$ and a refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$, the respective discrete solutions $w_{\ell+m}$ and w_ℓ satisfy*

$$\|\sigma_\ell - \sigma_{\ell+m}\|_0^2 \approx E_P(w_\ell) - E_P(w_{\ell+m}).$$

Proof. Due to $W_\ell \subset W_{\ell+m}$, we apply the same arguments as in the proof of Theorem 5.2 and Lemma 5.1 replacing w by $w_{\ell+m}$. □

For a refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$, $T \in \mathcal{T}_\ell$ and $E \in \mathcal{E}(T)$, the oscillations on the coarse cells and edges dominate the oscillations on the sub meshes

$$\mathcal{T}_{\ell+m}(T) := \{\hat{T} \in \mathcal{T}_{\ell+m} \mid \hat{T} \subset T\} \quad \text{and} \quad \mathcal{E}_{\ell+m}(E) := \{\hat{E} \in \mathcal{E}_{\ell+m}^N \mid \hat{E} \subset E\}.$$

In turn, the oscillations are dominated by the local error estimator $\eta_{P,\ell}(T)$. This is stated in the following lemma.

Lemma 5.4. *Let $\mathcal{T}_\ell$ be a mesh consisting of either triangles or quadrilaterals. For all $T \in \mathcal{T}_\ell$ there holds*

$$\operatorname{osc}^2(f_\Omega, \mathcal{T}_k(T)) \leq \operatorname{osc}^2(f_\Omega, T) \leq \eta^2_{P,\ell}(T),$$

and

$$\operatorname{osc}^2(f_N, \mathcal{E}_k(E)) \leq \operatorname{osc}^2(f_N, E) \leq \eta^2_{P,\ell}(T),$$

for all $E \in \mathcal{E}(T) \cap \mathcal{E}_\ell^N$.

Proof. We observe that $\tilde{f} \in \mathcal{P}_0(\mathcal{T}_k(T); \mathbb{R}^d)$ with $\tilde{f}_{|\hat{T}} := f_{\Omega,\hat{T}}$ and $\hat{T} \in \mathcal{T}_k(T)$ is the L^2-projection of f on $\mathcal{P}_0(\mathcal{T}_k(T); \mathbb{R}^d)$ and that $f_{\Omega,T} \in \mathcal{P}_0(\mathcal{T}_k(T); \mathbb{R}^d)$. Thus,

$$\begin{aligned}\operatorname{osc}^2(f_\Omega, \mathcal{T}_k(T)) &= \sum_{\hat{T} \in \mathcal{T}_k(T)} |\hat{T}| \|f_\Omega - f_{\Omega,\hat{T}}\|^2_{L^2(\hat{T};\mathbb{R}^d)} \\ &\leq |T| \sum_{\hat{T} \in \mathcal{T}_k(T)} \|f_\Omega - f_{\Omega,T}\|^2_{0,\hat{T}} = \operatorname{osc}^2(f_\omega, T).\end{aligned}$$

Since $f_{\Omega,T}$ is the L^2-projection of f_Ω in $\mathcal{P}_0(T; \mathbb{R}^d)$ and $\operatorname{div}(\sigma_\ell) \in \mathcal{P}_0(T; \mathbb{R}^d)$, we have

$$\operatorname{osc}^2(f_\Omega, T) \leq |T| \|f_\Omega + \operatorname{div}(\sigma_\ell)\|^2_{0,T} \leq \eta_{P,\ell}(T).$$

The second assertion follows by the same arguments. □

5.2 Convergence of the AFEM algorithm

A proof of the convergence of AFEM for the discrete scheme with affine and constant functions based on triangles can be found in [27, 40]. In next two section, we follow the arguments for linear elliptic problems found in [103, 42] and give an alternative proof. For a better understanding and the sake of completeness, we repeat and rearrange some of the original arguments. The starting point to show the convergence of the AFEM algorithm is to proof the reduction of the error estimator $\eta_{P,\ell}$ for increasing refinement level $\ell \in \mathbb{N}$. We denote the set of cells refined from level ℓ to level $\ell + m$ by

$$\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m} := \{T \in \mathcal{T}_\ell \mid T \notin \mathcal{T}_{\ell+m}\}$$

with $m \geq 1$ whereas the set of unrefined elements is given by $\mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$. Moreover, we define the usual patches $\omega_E := T \cup T'$ for $T, T' \in \mathcal{T}_\ell$ and $E \in \mathcal{E}(T) \cap \mathcal{E}(T')$ as well as $\omega_T := \bigcup_{E \in \mathcal{E}(T)} \omega_E$ and fix an unit normal n_E per edge. Note that the trace theorem and the shape regularity of $\mathcal{T}_\ell$ imply

$$\begin{aligned}\|[\sigma_{\ell+m} - \sigma_\ell] n_E\|_{0,E} &\lesssim |E|^{-1/2} \|\sigma_{\ell+m} - \sigma_\ell\|_{0,\omega_E} \text{ for } E \in \mathcal{E}_\ell^\circ, \\ \|(\sigma_{\ell+m} - \sigma_\ell) n_E\|_{0,E} &\lesssim |E|^{-1/2} \|\sigma_{\ell+m} - \sigma_\ell\|_{0,\omega_E} \text{ for } E \in \mathcal{E}_\ell^N.\end{aligned} \tag{5.6}$$

Moreover, for $a, b > 0$, we will repeatedly use the identity

$$(a+b)^2 = \min_{\lambda>0} \Big((1+\lambda)a^2 + (1+1/\lambda)b^2\Big). \tag{5.7}$$

In order to show the estimator reduction from the level ℓ to the level $\ell+m$ we consider the mesh $\mathcal{T}_{\ell+m}$ obtained after m additional iterations of the AFEM sequence (5.1). For a better overview, we split the mesh $\mathcal{T}_\ell$ into the cells which were refined $\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}$ and the ones which were not $\mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$. We begin with the following Lemma for the unrefined cells.

Lemma 5.5. *There exists a constant* $\Lambda_0 > 0$ *such that*

$$\eta^2_{P,\ell+m}(T) \le (1+\lambda)\eta^2_{P,\ell}(T) + \Lambda_0(1+1/\lambda)\|\sigma_{\ell+m} - \sigma_\ell\|^2_{L^2(\omega_T;\mathbb{R}^{d\times d})}$$

for all $\lambda > 0$ *and* $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$.

Proof. From the triangle inequality, we obtain

$$\begin{aligned}
|\eta_{P,\ell+m}(T) - \eta_{P,\ell}(T)| \le |T|^{1/4}\Big(&\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_\ell^\circ} \big(\|[\sigma_{\ell+m}]n_E\|_{L^2(E;\mathbb{R}^d)} - \|[\sigma_\ell]n_E\|_{L^2(E;\mathbb{R}^d)}\big)^2 \\
&+ \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_\ell^N} \big(\|f_N - \sigma_{\ell+m}n_E\|_{L^2(E;\mathbb{R}^d)} - \|f_N - \sigma_\ell n_E\|_{L^2(E;\mathbb{R}^d)}\big)^2\Big)^{1/2} \\
\le |T|^{1/4}\Big(&\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_\ell^\circ} \|[\sigma_{\ell+m} - \sigma_\ell]n_E\|^2_{L^2(E;\mathbb{R}^d)} \\
&+ \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_\ell^N} \|(\sigma_\ell - \sigma_{\ell+m})n_E\|^2_{L^2(E;\mathbb{R}^d)}\Big)^{1/2}
\end{aligned}$$

The assertion follows from (5.6) and (5.7). □

Next, we derive the bound for the estimator $\eta_{\ell+m}$ on the cells which were refined.

Lemma 5.6. *There exists a constant* $\Lambda_1 > 0$ *such that*

$$\eta^2_{P,\ell+m}(\mathcal{T}_{\ell+m}(T)) \le 2^{-1/2}(1+\lambda)\eta^2_{P,\ell}(T) + \Lambda_1(1+1/\lambda)\|\sigma_{\ell+m} - \sigma_\ell\|^2_{0,\omega_T}.$$

for all $\lambda > 0$ *and* $T \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}$.

Proof. Due to the refinement patterns used in the AFEM algorithm (for triangles see Figure 5.1), we have $|\hat{T}| \le |T|/2$ for all $\hat{T} \in \mathcal{T}_{\ell+m}(T)$. Thus, we obtain from

(5.7),

$$
\begin{aligned}
\eta^2_{P,\ell+m}(\mathcal{T}_{\ell+m}(T)) = &\sum_{\hat{T}\in\mathcal{T}_{\ell+m}(T)} \Big(|\hat{T}|\|f_\Omega\|^2_{0,\hat{T}} \\
&+ |\hat{T}|^{1/2}\big(\sum_{\hat{E}\in\mathcal{E}^\circ_{\ell+m}(\hat{T})} \|[\sigma_{\ell+m}]n_{\hat{E}}\|^2_{0,\hat{E}} + \sum_{\hat{E}\in\mathcal{E}^N_{\ell+m}(\hat{T})} \|(f_N-\sigma_{\ell+m})n_{\hat{E}}\|^2_{0,\hat{E}}\big)\Big) \\
\le\ & |T|/2 \sum_{\hat{T}\in\mathcal{T}_{\ell+m}(T)} \|f_\Omega\|^2_{0,\hat{T}} \\
&+ (1+1/\lambda) \sum_{\hat{T}\in\mathcal{T}_{\ell+m}(T)} |\hat{T}|^{1/2}\Big(\sum_{\hat{E}\in\mathcal{E}^\circ_{\ell+m}(\hat{T})} \|[\sigma_{\ell+m}-\sigma_\ell]n_{\hat{E}}\|^2_{0,\hat{E}} \\
&+ \sum_{\hat{E}\in\mathcal{E}^N_{\ell+m}(\hat{T})} \|(\sigma_{\ell+m}-\sigma_\ell)n_{\hat{E}}\|^2_{0,\hat{E}}\Big) \\
&+ (|T|/2)^{1/2}(1+\lambda) \sum_{E\in\mathcal{E}^\circ_\ell(T)} \|[\sigma_\ell]n_E\|^2_{0,E} + \sum_{E\in\mathcal{E}^N_\ell(T)} \|(f_N-\sigma_\ell)n_E\|^2_{0,E}
\end{aligned}
$$

The assertion follows from the definition of $\eta^2(T)$ and (5.6). □

Remark 5.7. We note that $|\hat{K}| \le |K|/4$ for all $\hat{K} \in \mathcal{T}_{\ell+m}(K)$ for the proposed refinement of a mesh consisting of only quadrilaterals. In this case, the estimate of the previous Lemma reads

$$\eta^2_{\ell+m}(\mathcal{T}_{\ell+m}(T)) \le 4^{-1/2}(1+\lambda)\eta^2_{P,\ell}(T) + \Lambda_1(1+1/\lambda)\|\sigma_{\ell+m}-\sigma_\ell\|^2_{0,\omega_T}.$$

The following result is just the combination of the previous two lemmas.

Lemma 5.8. *There exists a constant $\Lambda > 0$ which only depends on the initial mesh $\mathcal{T}_0$ such that the estimators $\eta_{P,\ell}$ and $\eta_{\ell+m}$ satisfy*

$$\eta_{P,\ell+m} \le \Big(\eta^2_{P,\ell}(\mathcal{T}_\ell\cap\mathcal{T}_{\ell+m}) + 2^{-1/2}\eta^2_{P,\ell}(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})\Big)^{1/2} + \Lambda\|\sigma_{\ell+m}-\sigma_\ell\|_0.$$

Proof. From Lemma 5.5 and 5.6, we obtain

$$
\begin{aligned}
\eta^2_{\ell+m} &= \eta^2_{\ell+m}(\mathcal{T}_\ell\cap\mathcal{T}_{\ell+m}) + \eta^2_{\ell+m}(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}) \\
&\le (1+\lambda)\Big(\eta^2_\ell(\mathcal{T}_\ell\cap\mathcal{T}_{\ell+m}) + 2^{-1/2}\eta^2_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})\Big) \\
&\quad + 4(1+1/\lambda)\max\{\Lambda_0,\Lambda_1\}\|\sigma_{\ell+m}-\sigma_\ell\|^2_0.
\end{aligned}
$$

The assertion directly follows from (5.7). □

Eventually, the estimator reduction is a direct consequence of the Dörfler marking (5.2) and the above result.

Theorem 5.9. *Given the bulk parameter $0 < \theta \leq 1$ and the constant $\Lambda > 0$ from Theorem 5.8, then $\rho_1 := (1 - \theta + 2^{-1/2}\theta)^{1/2} < 1$ satisfies*

$$\eta_{P,\ell+1} \leq \rho_1 \eta_{P,\ell} + \Lambda \|\sigma_{\ell+1} - \sigma_\ell\|_0.$$

Proof. The Dörfler marking (5.2) implies

$$\theta \eta_{P,\ell}^2 \leq \eta_{P,\ell}^2(\mathcal{M}_\ell) \leq \eta_{P,\ell}^2(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}) = \eta_{P,\ell}^2 - \eta_{P,\ell}^2(\mathcal{T}_\ell \cap \mathcal{T}_{\ell+m})$$

which gives

$$\eta_{P,\ell}^2(\mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}) \leq (1-\theta)\eta_{P,\ell}^2.$$

Thus, Lemma 5.8 yields

$$\begin{aligned}
\eta_{P,\ell+1} &\leq \Big((1-\theta)\eta_{P,\ell}^2 + 2^{-1/2}(\eta_{P,\ell}^2 - \eta_{P,\ell}(\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1})\Big)^{1/2} + \Lambda\|\sigma_{\ell+1} - \sigma_\ell\|_0 \\
&\leq \Big((1-\theta)\eta_{P,\ell}^2 + 2^{-1/2}(\eta_{P,\ell}^2 - (1-\theta)\eta_{P,\ell}^2)\Big)^{1/2} + \Lambda\|\sigma_{\ell+1} - \sigma_\ell\|_0 \\
&= \rho_1 \eta_{P,\ell} + \Lambda\|\sigma_{\ell+1} - \sigma_\ell\|_0.
\end{aligned}$$

□

The estimator reduction of the previous theorem yields the contraction of the the weighted sum

$$\xi_\ell^2 := \eta_{P,\ell}^2 + \beta\delta_\ell$$

with $\delta_\ell := E_P(w_\ell) - E_P(w)$ and $\beta \geq 0$. Obviously, the convergence of the AFEM algorithm is a direct consequence of the contraction property of this sum.

Theorem 5.10. *There exist parameters $\beta \geq 0$ and $0 < \rho_2 < 1$ such that*

$$\xi_{\ell+1} \leq \rho_2 \xi_\ell$$

for all $\ell \in \mathbb{N}_0$.

Proof. Theorem 5.9 and (5.7) imply

$$\eta_{P,\ell+1}^2 \leq (1+\lambda)\rho_1^2\eta_{P,\ell}^2 + (1+1/\lambda)\Lambda^2\|\sigma_{\ell+1} - \sigma_\ell\|_0^2$$

for $0 < \lambda < \rho_1^{-2} - 1$. Corollary 5.3 implies

$$\eta_{P,\ell+1}^2 \leq \rho_\lambda \eta_{P,\ell}^2 + \beta_\lambda(E_P(w_\ell) - E_P(w_{\ell+1})) = \rho_\lambda\eta_{P,\ell}^2 + \beta_\lambda\delta_\ell - \beta_\lambda\delta_{\ell+1}$$

with $\rho_\lambda := (1+\lambda)\rho_1^2 < 1$ and a further constant $\beta_\lambda \geq 0$ which also depends on λ. From Theorems 5.2 and 4.1, we conclude that there is a constant $C > 0$ such that $\delta_\ell \leq C\eta_{P,\ell}^2$. With

$$\vartheta := \frac{(1-\rho_\lambda)\beta_\lambda}{\beta_\lambda C + 1}, \quad \rho_2 := \rho_\lambda + \vartheta C < 1$$

we obtain

$$\eta_{P,\ell+1}^2 + \beta_\lambda \delta_{\ell+1} \leq \rho_\lambda \eta_{P,\ell}^2 + \beta_\lambda \delta_\ell \leq \rho_2 \eta_{P,\ell}^2 + (\beta_\lambda - \vartheta)\delta_\ell = \rho_2(\eta_{P,\ell} + \beta_\lambda \delta_\ell).$$

□

5.3 Optimal convergence of the AFEM algorithm

In this section, we restrict the discretization to the one based on triangles and thus are able to use the standard arguments proposed in [103, 42]. The convergence rate of the AFEM algorithm is usually described through the introduction of approximation classes. For this purpose, we set

$$W_{\mathcal{T}} := \mathcal{P}_1(\mathcal{T};\mathbb{R}^d) \times \mathcal{P}_0(\mathcal{T},\mathbb{R}^{d\times d}_{\mathrm{sym,dev}}) \times \mathcal{P}_0(\mathcal{T})$$

for a triangulation $\mathcal{T}$ and denote the minimizer of E_P over $W_{\mathcal{T}}$ by $w_{\mathcal{T}} \in W_{\mathcal{T}}$. Given $s > 0$, the approximation class $\mathcal{A}_s$ is defined as

$$\mathcal{A}_s := \{(w, f_\Omega, f_N) \in W \times L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_N;\mathbb{R}^d) \mid |(w, f_\Omega, f_N)|_{\mathcal{A}_s} < \infty\}$$

with

$$|(w, f_\Omega, f_N)|_{\mathcal{A}_s} := \sup_{N\in\mathbb{N}} N^s \min_{\substack{\mathcal{T}\in\mathbb{T}\\ |\mathcal{T}|-|\mathcal{T}_0|\leq N}} \Big(\mathrm{osc}^2(f_\Omega,\mathcal{T}) + \mathrm{osc}^2(f_N,\mathcal{E}^N) + E_P(w_{\mathcal{T}}) - E_P(w)\Big)^{1/2}.$$

The first step to prove optimality in the sense of approximation classes is to establish the discrete reliability of the error estimator.

Theorem 5.11. *For a refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$, the estimator fulfills*

$$E_P(w_\ell) - E_P(w_{\ell+m}) \lesssim \|\sigma_{\ell+m} - \sigma_\ell\|_0^2 \lesssim \eta_{P,\ell}^2(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}).$$

Proof. We define a $v_\ell \in V_\ell$ with $u_\ell - u_{\ell+m} - v_\ell = 0$ on $\mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$ and as the Scott-Zhang interpolation of $u_\ell - u_{\ell+m}$ on $\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}$. Theorem 5.2 and $(\sigma_{\ell+m} - \sigma_\ell, \varepsilon(v_\ell))_0 = 0$ imply

$$\begin{aligned} \|\sigma_{\ell+m} - \sigma_\ell\|_0 &\lesssim (\sigma_{\ell+m} - \sigma_\ell, \varepsilon(u_{\ell+m} - u_\ell - v_\ell))_0 \\ &= (f_\Omega, u_\ell - u_{\ell+m} - v_\ell)_0 - (\sigma_\ell, \varepsilon(u_\ell - u_{\ell+m} - v_\ell))_0 \end{aligned}$$

This and other arguments from [42, 103] prove the assertion. Since the remaining details are the same for linear problems, they are omitted here. □

Assume that $(w, f_\Omega, f_N) \in \mathcal{A}_s$ and choose a minimal $N_\ell \in \mathbb{N}$ such that

$$|(w, f_\Omega, f_N)|_{\mathcal{A}_s} \leq \tau \xi_\ell N_\ell^s$$

for a given $\tau > 0$ and $\ell \in \mathbb{N}$. Evidently, for such a minimal N_ℓ there holds

$$N_\ell \leq 2(N_\ell - 1) \leq 2|(w, f_\Omega, f_N)|_{\mathcal{A}_s}^{1/s}(\tau\xi_\ell)^{-1/s}. \tag{5.8}$$

The definition of the approximation class $\mathcal{A}_s$ implies the existence of triangulations $\tilde{\mathcal{T}}_\ell \in \mathbb{T}$ and a discrete solution $\tilde{w}_\ell$ such that

$$|\tilde{\mathcal{T}}_\ell| - |\mathcal{T}_0| \leq N_\ell$$

and

$$E_P(\tilde{w}_\ell) - E_P(w) + \mathrm{osc}^2(f, \tilde{\mathcal{T}}_\ell) + \mathrm{osc}^2(g, \tilde{\mathcal{E}}_\ell^N) \leq N_\ell^{-2s}|(w, f_\Omega, f_N)|_{\mathcal{A}_s}^2 \leq (\tau\xi_\ell)^2. \tag{5.9}$$

The use of the newest vertex bisections provides the existence of a unique refinement $\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell$ which is a refinement of $\mathcal{T}_\ell$ and $\tilde{\mathcal{T}}_\ell$, and of minimal cardinality. It is called the overlay of $\mathcal{T}_\ell$ and $\tilde{\mathcal{T}}_\ell$ and satisfies

$$|\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell| - |\mathcal{T}_\ell| \leq |\tilde{\mathcal{T}}_\ell| - |\mathcal{T}_0| \leq N_\ell.$$

Moreover, we observe

$$\begin{aligned} |\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)| \leq \sum_{T \in \mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)} (|(\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)(T)| - 1) &= |(\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell) \setminus \mathcal{T}_\ell| - |\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)| \\ &= |\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell| - |\mathcal{T}_\ell|. \end{aligned}$$

Thus, we conclude

$$|\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)| \leq N_\ell. \tag{5.10}$$

Lemma 5.12. *There holds*

$$\eta_{P,\ell} \lesssim \eta_{P,\ell}(\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)).$$

Proof. From Theorem 4.1 and 5.11 we have

$$\begin{aligned} \eta_{P,\ell}^2 &\lesssim \|\sigma - \sigma_\ell\|_0^2 + \mathrm{osc}^2(f_\Omega, \mathcal{T}_\ell) + \mathrm{osc}^2(f_N, \mathcal{E}_\ell^N) \\ &\lesssim \|\hat{\sigma}_\ell - \sigma_\ell\|_0^2 + \|\sigma - \hat{\sigma}_\ell\|_0^2 + \mathrm{osc}^2(f_\Omega, \mathcal{T}_\ell) + \mathrm{osc}^2(f_N, \mathcal{E}_\ell^N) \\ &\lesssim \eta_{P,\ell}^2(\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)) + E_P(\hat{w}_\ell) - E_P(w) + \mathrm{osc}^2(f_\Omega, \mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell) + \mathrm{osc}^2(f_N, \hat{\mathcal{E}}_\ell^N). \end{aligned}$$

where $\hat{\mathcal{E}}_\ell^N$ is the set of all edges of $\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell$ on Γ_N and $\hat{\sigma} := \sigma(\hat{u}_\ell, \hat{p}_\ell)$ with the discrete solution $\hat{w}_\ell := (\hat{u}_\ell, \hat{p}_\ell, \hat{\alpha}_\ell) \in W_{\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell}$. Lemma 5.4 and (5.9) yield

$$\begin{aligned} E_P(\hat{w}_\ell) - E_P(w) + \mathrm{osc}^2(f_\Omega, \mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell) + \mathrm{osc}^2(f_N, \hat{\mathcal{E}}_\ell^N) \leq \\ E_P(\tilde{w}_\ell) - E_P(w) + \mathrm{osc}^2(f_\Omega, \tilde{\mathcal{T}}_\ell) + \mathrm{osc}^2(f_N, \tilde{\mathcal{E}}_\ell^N) \\ \leq (\tau\xi_\ell)^2 \lesssim \tau^2 \eta_{P,\ell}^2 \end{aligned}$$

with the discrete solution $\tilde{w}_\ell \in W_{\tilde{\mathcal{T}}_\ell}$ and the set $\tilde{\mathcal{E}}_\ell^N$ of all edges of $\tilde{\mathcal{T}}_\ell$ on Γ_N. Thus, the assertion follows with a sufficiently small $\tau > 0$. □

An import ingredient to prove the optimal convergence of the AFEM algorithm is the BDV-Theorem introduced in [16].

Theorem 5.13. *Let $\mathcal{M}_\ell$ with $\ell \in \mathbb{N}$ be the set of elements marked by the AFEM algorithm, i.e. marked by bulk criterion and closure. It holds*

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq |\mathcal{T}_\ell \setminus \mathcal{T}_0| \lesssim |\mathcal{M}_0 \cup \cdots \cup \mathcal{M}_{\ell-1}|$$

where the constant in the second estimate solely depends on $\mathcal{T}_0$.

Proof. See proof of Theorem 2.4 in [16]. □

Theorem 5.14. *There exist a bulk parameter $0 < \theta_0 \leq 1$ and a constant $C(s) > 0$ such that for all bulk parameters $0 < \theta \leq \theta_0$ of the AFEM algorithm it holds*

$$(|\mathcal{T}_\ell|-|\mathcal{T}_0|)^s(E_P(w_\ell)-E_P(w)+\mathrm{osc}^2(f_\Omega,\mathcal{T}_\ell)+\mathrm{osc}^2(f_N,\mathcal{E}_\ell^N))^{1/2} \leq C(s)|(w,f_\Omega,f_N)|_{\mathcal{A}_s}.$$

Proof. From Lemma 5.12 there exists $0 < \theta_0 \leq 1$ so that $\theta_0 \eta_{P,\ell}^2 \leq \eta_{P,\ell}^2(\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell))$. This means that $\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)$ also satisfies the bulk criterion (5.2) for all bulk parameters $0 < \theta \leq \theta_0$. Thus, since $\mathcal{M}_\ell$ in Theorem 5.13 is of minimal cardinality, we obtain from (5.8) and (5.10),

$$|\mathcal{M}_\ell| \leq |\mathcal{T}_\ell \setminus (\mathcal{T}_\ell \oplus \tilde{\mathcal{T}}_\ell)| \leq N_\ell \leq 2|(w,f_\Omega,f_N)|_{\mathcal{A}_s}^{1/s}(\tau\xi_\ell)^{-1/s}.$$

and from Theorem 5.13

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \sum_{k=0}^{\ell-1} |\mathcal{M}_k| \lesssim |(w,f_\Omega,f_N)|_{\mathcal{A}_s}^{1/s}\tau^{-1/s}\sum_{k=0}^{\ell-1}\xi_k^{-1/s}.$$

Theorem 5.10 with $0 < \rho_2 < 1$ yields $\xi_\ell \leq \rho_2^{\ell-k}\xi_k$ for $0 \leq k \leq \ell$. Hence, we obtain

$$\sum_{k=0}^{\ell-1}\xi_k^{-1/s} \leq \xi_\ell^{-1/s}\sum_{k=0}^{\ell-1}\rho_2^{(\ell-k)/s} \leq \xi_\ell^{-1/s}\frac{\rho_2^{1/s}}{1-\rho_2^{1/s}}.$$

and, therefore,

$$\xi_\ell(|\mathcal{T}_\ell| - |\mathcal{T}_0|)^s \lesssim \frac{\rho_2}{\tau(1-\rho_2^{1/s})^s}|(w,f_\Omega,f_N)|_{\mathcal{A}_s}.$$

The definition of ξ_ℓ as well as Theorem 5.2 and 4.1 eventually yield the assertion. □

The optimal convergence in terms of the energy E_P implies the convergence of stresses.

Corollary 5.15. *Under the assumption of Theorem 5.14, there exists a constant* $\bar{C}(s) >$ *such that*

$$(|\mathcal{T}_\ell| - |\mathcal{T}_0|)^s (\|\sigma - \sigma_\ell\|_0^2 + \mathrm{osc}^2(f_\Omega, \mathcal{T}_\ell) + \mathrm{osc}^2(f_N, \mathcal{E}_\ell^N))^{1/2} \leq \bar{C}(s) |(w, f, g)|_{\mathcal{A}_s}.$$

Proof. The assertion directly follows from Theorem 5.2 and 5.14. □

Remark 5.16. We observe that all arguments except Theorem 5.13 and (5.10) still hold if the mesh consists of quadrilaterals. However, it remains an open question how to control the number of additionally marked elements within a closure algorithm.

6 Numerical results

In this chapter, we present some numerical examples to illustrate the theoretical results. We start with experiments concerned with the elastoplastic problem without contact conditions of Section 1.3. This is followed by numerical results for the frictional contact problem of Section 1.5. The results of the first example have also been published in [97].

We briefly introduce Uzawa's algorithm for nonlinear problems as known from [52, 55]. We denote the number of degrees of freedom of V_ℓ and Q_ℓ by n and m, respectively. The basis functions of V_ℓ are denoted by φ_j with $j = 1, \ldots, n$. With scalar valued Lagrange polynomials ϕ_j, $j = 1, \ldots, m/2$ associated to the nodes of the Gauss quadrature $\mathcal{G}^d$, we set

$$\psi_\iota = \begin{pmatrix} \phi_j & 0 \\ 0 & -\phi_j \end{pmatrix} \text{ and } \psi_{\tilde{\iota}} = \begin{pmatrix} 0 & \phi_j \\ \phi_j & 0 \end{pmatrix},$$

where ι, $\tilde{\iota}$ are an appropriate numbering of the degrees of freedom. The functions Φ_k, $k = 1, \ldots, m_C$ denote the basis of the space $M_{\mathscr{L}}$ of Lagrange Multipliers for the contact conditions and by $\tilde{\Phi}_k$, $k = 1, \ldots, m_F$ we denote the basis of $M_{\mathscr{L}}^{d-1}$.

For the elastoplasticty with linear kinematic hardening, we define matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ and $C, D \in \mathbb{R}^{m\times m}$ as

$$\begin{aligned} A_{ij} &:= (\mathbb{C}\varepsilon(\varphi_j), \varepsilon(\varphi_i))_0, \quad & B_{ij} &:= (-\mathbb{C}\psi_j, \varepsilon(\varphi_i))_0, \\ C_{ij} &:= ((\mathbb{C} + \mathbb{H})\psi_j, \psi_i)_0, \quad & D_{ij} &:= \sigma_y(\psi_j, \psi_i)_0. \end{aligned}$$

The discretization of the contact conditions yields matrices

$$D_{C,ij} := (\Phi_j, \varphi_{i,n})_{0,\Gamma_C} \text{ and } D_{C,ij} := (\tilde{\Phi}_j, f_F \varphi_{i,t})_{0,\Gamma_C}.$$

Moreover, we introduce the right hand side

$$L_i := (f, \varphi_i)_0 + (g, \gamma_{|\Gamma_N}(\varphi_i))_{0,\Gamma_N}.$$

Obviously, the matrices A, C and D are symmetric and positive definite from the properties of the bilinear form. Additionally, the matrices C and D are diagonal matrices due to the properties of the Lagrange basis.

Let $\bar{\Lambda}_{P,\ell} := \{z \in \mathbb{R}^m \mid \sum_{j=1}^m z_j\psi_j \in \Lambda_{P,\ell}\}$. Hence, the discrete mixed formulation of elastoplasticity with linear kinematic hardening is equivalent to find $(x, y, z) \in$

$\mathbb{R}^n \times \mathbb{R}^m \times \bar{\Lambda}_{P,\ell}$, such that,

$$\begin{aligned} &Ax + By = L, \\ &B^\top x + Cy + Dz = 0, \\ &(z - \tilde{z})^\top Dy \leq 0 \end{aligned}$$

for all $\tilde{z} \in \bar{\Lambda}_{P,\ell}$.

Let

$$\bar{\Lambda}_{C,\mathscr{L}} := \{z \in \mathbb{R}^{m_C} \mid \sum_{j=1}^{m_C} z_j \psi_j \in \Lambda_{C,\mathscr{L}}\}$$

and

$$\bar{\Lambda}_{F,\mathscr{L}} := \{z \in \mathbb{R}^{m_F} \mid \sum_{j=1}^{m_F} z_j \psi_j \in \Lambda_{F,\mathscr{L}}\}.$$

Moreover, we set $\tilde{g}_k := (\Phi_k, g)_{0,\Gamma_C}$, $k = 1, \ldots, m_C$. Hence, the discrete mixed formulation of the frictional contact problem in elastoplasticity with linear kinematic hardening is equivalent to find $(x, y, z, z_C, z_F) \in \mathbb{R}^n \times \mathbb{R}^m \times \bar{\Lambda}_{P,\ell} \times \bar{\Lambda}_{C,\mathscr{L}} \times \bar{\Lambda}_{F,\mathscr{L}}$, such that,

$$\begin{aligned} &Ax + By + D_C z_C + D_F z_F = L, \\ &B^\top x + Cy + Dz = 0, \\ &(z - \tilde{z})^\top Dy \leq 0 \\ &(z_C - \tilde{z}_C)^\top (D_C^\top x - \tilde{g}) \leq 0 \\ &(z_F - \tilde{z}_F)^\top D_F^\top x \leq 0 \end{aligned}$$

for all $\tilde{z} \in \bar{\Lambda}_{P,\ell}$, $\tilde{z}_C \in \bar{\Lambda}_{C,\mathscr{L}}$ and $\tilde{z}_F \in \bar{\Lambda}_{F,\mathscr{L}}$.

Let $P : \mathbb{R}^m \to \bar{\Lambda}_{P,\ell}$,$P_C : \mathbb{R}^{m_C} \to \bar{\Lambda}_{C,\mathscr{L}}$, and $P_F : \mathbb{R}^{m_F} \to \bar{\Lambda}_{F,\mathscr{L}}$ be projections onto $\bar{\Lambda}_{P,\ell}$, $\bar{\Lambda}_{C,\mathscr{L}}$, and $\bar{\Lambda}_{F,\mathscr{L}}$, respectively. Furthermore, let $S \in \mathbb{R}^{(n+m)\times(n+m)}$ be an invertible matrix. For some parameters $\rho_1, \rho_2 > 0$, we define the iterative method

$$\begin{aligned} \begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} &= \begin{pmatrix} x^k \\ y^k \end{pmatrix} - \rho_1 S \begin{pmatrix} Ax^k + By^k + D_C z_C^k + D_F z_F^k - L \\ B^\top x^k + Cy^k + Dz^k \end{pmatrix}, \\ z^{k+1} &= P(z^k + \rho_2 D y^{k+1}), \\ z_C^{k+1} &= P_C(z_C^k + \rho_2 {D_C}^\top x^{k+1}), \\ z_F^{k+1} &= P_F(z_F^k + \rho_2 {D_F}^\top x^{k+1}). \end{aligned} \tag{6.1}$$

The convergence of x^k, y^k, z^k, z_C^k and z_F^k to x, y, z, z_C and z_F, respectively, is given in [52]. The iteratitive scheme results in the usual Uzawa's algorithm with projection if we chose the matrix S as

$$S := \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}^{-1}.$$

In the case of the inexact Uzawa's algorithm, the matrix S is choosen as an appropriate approximation of the inverse or replaced by some steps of an iterative solution scheme for systems of linear equations, cf. [43]. Moreover, we note that the condition numbers of the matrices B, S, C and D strongly depend on difference in the size of cells of the same mesh and increases with adaptive refinements.

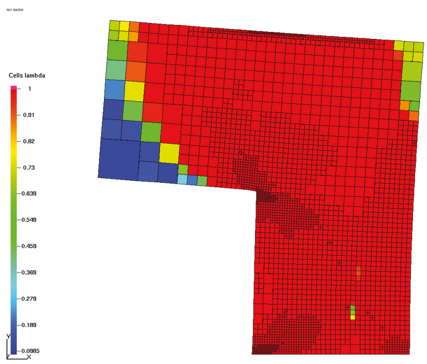
6.1 Elastoplasticity

In this section we investigate some of the results on the discrete approximation of the variational inequality (1.14) and the saddle point problem (1.17). As we have seen the two formulations are equivalent in infinite dimensions. Moreover we observed that the discrete versions are equivalent if the set of discrete Lagrange multipliers is conform. Throughout this section the approximations are computed from the mixed formulation with the help of Uzawa's method. Although this method is not very efficient it is easy to implement and that is why we chose it here. For an overview on more efficient solution algorithms of the discrete variational inequality, we refer to [44, 7, 65, 53] for modified Newton's methods and to [57, 102, 101] for algorithms of predictor-corrector type.

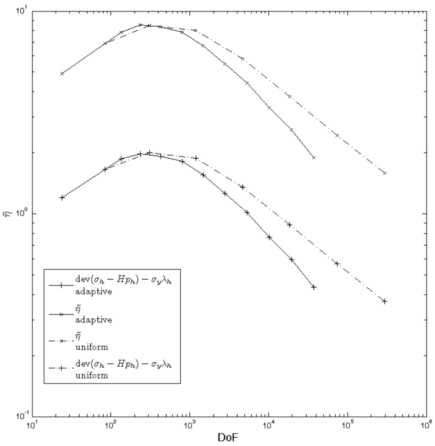
It is easy to construct a suitable projection due the choice of the Lagrange basis and the definition of the set of discrete multipliers. In every Gauss point, only two basis functions do not equal zero. Hence, we can easily compute the norm of a discrete function $\mu = \sum_{j=1}^{m} z_j \psi_j$ in every Gauss point from the coefficients of this two basis functions. Whenever the value of the norm is greater than one we simply divide the coefficients by the value. In this way, we ensure $\mu \in \Lambda_{P,\ell}$ and therefore $z \in \bar{\Lambda}_{P,\ell}$.

We use the estimators of Section 4.1 to define an adaptive finite element method like in Chapter 5. We use meshes of squares for the first examples as well as piecewise bilinear and piecewise constant functions for the displacement and the plastic strain, respectively. In order to determine the cells that shall be refined, we employ a so-called fixed fraction marking strategy. In such a strategy a fixed fraction of cells with the largest contributions to the overall error estimate is refined. The cells are refined by a division into four congruent squares and we allow for hanging nodes. The remaining parts of the AFEM algorithm remain the same as before.

In the first examples we use the L-shape domain $\Omega = (0,1)^2 \backslash (0,0.5)^2$ with homogeneous Dirichlet boundary conditions on $\Gamma_D := [0.5,1] \times \{0\}$. We set the surface traction to $f_N := 1.25$ on $[0,1] \times \{1\}$ and zero elsewhere as well as the volume force to $f_\Omega := 0$ everywhere. The material parameters are $\lambda_{\mathbb{C}} := 1000$, $\mu_{\mathbb{C}} = 1000$, $\xi := 100$ and $\sigma_y = 1.25$. Even though, the exact solution for this problem is not known, we expect singular behavior at the reentrant corner and at the points where the boundary conditions change. Indeed, we observe adaptive refinements towards those points as we can see in Figure 6.1a. Moreover, we see that the adaptive refinement yields greater convergence rates than an uniform refinement, as expected, see Figure 6.1b. We note that the term $\|\operatorname{dev}(\sigma_\ell - \mathbb{H}p - \ell) - \sigma_y \lambda_\ell\|$ is of the same



(a)



(b)

Figure 6.1: **(a)** Adaptive refinements, the colors indicate $(\lambda_\ell : \lambda_\ell)^{1/2}$. **(b)** Estimated convergence rates for adaptive and uniform refinements.

order as the rest of the estimator.

An important issue in the simulation of elastoplastic material behavior is the determination of zones of plastic deformation. The mixed scheme and Uzawa's method make this easier as they can help to better detect zones where only elastic deformations occur. The complement of the elastic region is the zone of possible plastic deformation. If plastic deformation actually occur depends on the complementary condition induced by the normality rule, see [101]. A purely elastic deformation is characterized by $p = 0$. However, for an iterative numerical scheme with a stopping criterion, it is not clear whether the numerical solution does actually approximate zero or just a small value. If the scheme is additionally based upon a regularization it becomes even harder to determine where the plastic strain is approximately zero. We denote the results of the numerical algorithms with stopping criterions by a tilde. In Figure 6.2, $(\tilde{p}_\ell : \tilde{p}_\ell)^{1/2}$ is depicted in several ranges and for different tolerances

$$(|x^{k+1} - x^k| + |y^{k+1} - y^k| + |z^{k+1} - z^k|)/(|x^k| + |y^k| + |z^k|) < \mathrm{tol}$$

with $\mathrm{tol} = 10^{-5}$ (Figure 6.2a-c) and $\mathrm{tol} = 10^{-10}$ (Figure 6.2d-f). As expected, the tolerance for the stopping criterion significantly influences the plastic variable close to zero. For $\mathrm{tol} = 10^{-5}$ the zones are not determinable.

We observe that $p_\ell = 0$ if $\lambda_{P,\ell} : \lambda_{P,\ell} < 1$, cf. [97]. Thus, we can use the Lagrange multiplier to determine the regions of elastic deformations. Figure 6.3 depicts the Lagrange multiplier for the same experiment as in Figure 6.2. Moreover, we use the same tolerances $\mathrm{tol} = 10^{-5}$ (Figure 6.3a-c) and $\mathrm{tol} = 10^{-10}$ (Figure 6.3d-f). In

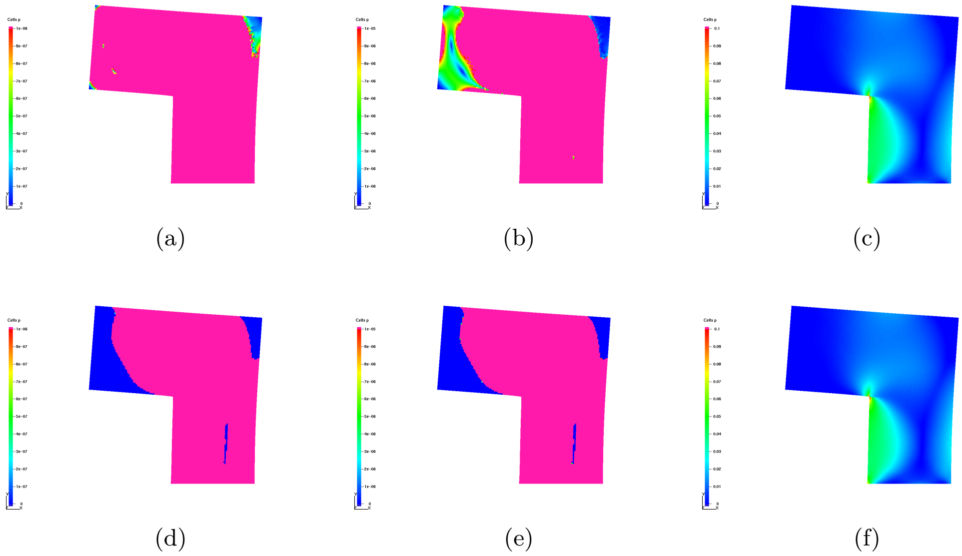


Figure 6.2: $(\tilde{p}_\ell : \tilde{p}_\ell)^{1/2}$ for different ranges and tolerances, **(a)** $[0, 10^{-6}]$, **(b)** $[0, 10^{-5}]$, **(c)** $[0, 10^{-1}]$ with $\mathrm{tol} = 10^{-5}$, and **(d)** $[0, 10^{-6}]$, **(e)** $[0, 10^{-5}]$, **(f)** $[0, 10^{-1}]$ with $\mathrm{tol} = 10^{-10}$.

contrast to the plastic strain, the quantity $\lambda_{P,\ell} : \lambda_{P,\ell}$ already gives a sharp criterion for $\mathrm{tol} = 10^{-5}$.

As mentioned in Remark 1.6, the linear kinematic hardening model can easily be extended to multiple yield surfaces. In the next example we introduce a second yield surface. The yield and hardening parameters are given as $\sigma_{y,0} := 1.25$, $\sigma_{y,1} := 5$, $\xi_0 := 100$, and $\xi_1 := 50$. All other material constants, the exterior forces, boundary conditions and the domain remain the same. Figure 6.4 shows the norm of Lagrange multipliers $\lambda_{0,P,\ell}$ and $\lambda_{1,P,\ell}$ which describe the first and second yield surface, respectively.

In the next numerical experiment, we focus on the influence of the hardening parameter ξ. We change the discretization to the one of Chapter 5 based on triangles and decrease the surface traction to $f_N = 0.75$. Figure 6.5 shows the norm of the discrete Lagrange multiplier for different values of ξ. The meshes result from 30 steps of the AFEM loop with bulk marking. We observe that for small hardening parameters the deformation and the size of plastic zone increases significantly. This is the reason why we have chosen a smaller surface traction. The variation of the hardening tensor also has a big influence on the convergence of the AFEM algorithm. For small values of ξ the model comes close to the case of perfect plasticity and the solvability of the discrete scheme decreases, see Figure 6.6.

Next, we investigate the dependence of the convergence rate on the bulk parameter. Apart from a different yield stress $\sigma_y = 1.25$, we choose the same setting as in

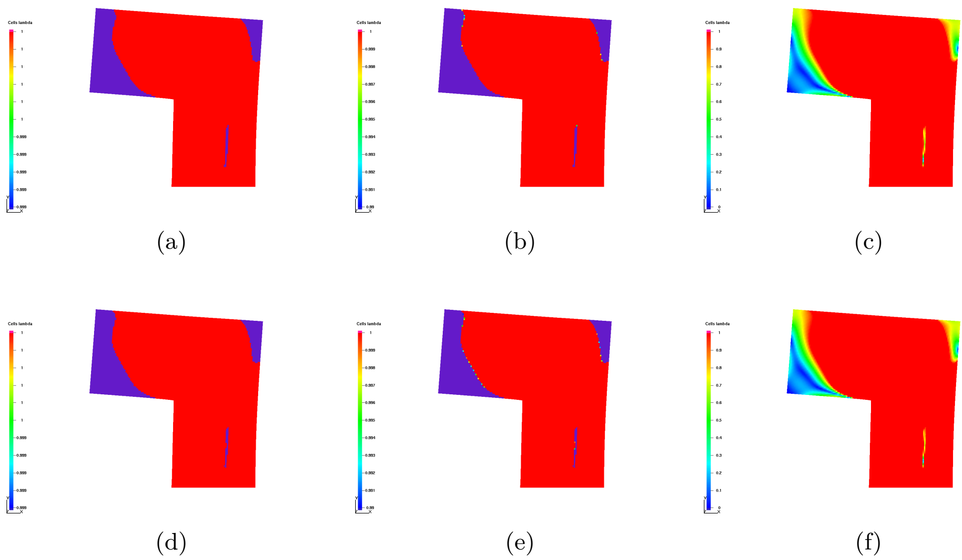
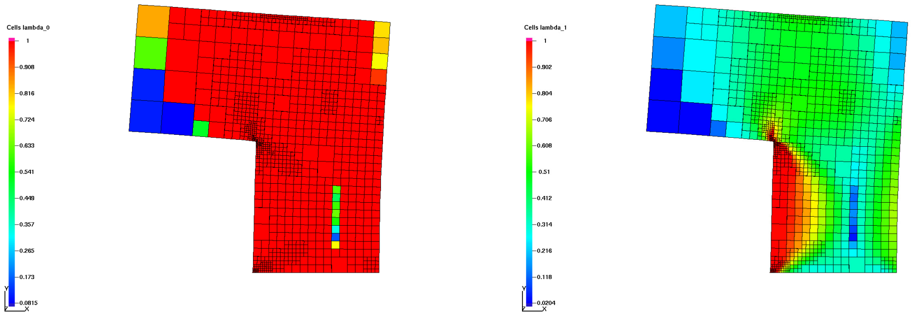


Figure 6.3: $(\tilde{\lambda}_{P,\ell} : \tilde{\lambda}_{P,\ell})^{1/2} \approx (\lambda_{P,\ell} : \lambda_{P,\ell})^{1/2}$ for different ranges and tolerances, **(a)** $[0.999, 1]$, **(b)** $[0.99, 1]$, **(c)** $[0, 1]$ with $\mathrm{tol} = 10^{-5}$, and **(d)** $[0.999, 1]$, **(e)** $[0.99, 1]$, **(e)** $[0, 1]$ with $\mathrm{tol} = 10^{-10}$.

the previous example with hardening parameter $\xi = 100$. Figure 6.7 shows the error estimator over the degrees of freedom for different parameters. We observe that for bulk parameters smaller than 0.5 the convergence rate is asymptotically the same whereas for greater values the convergence slows down.

The next experiments are concerned with the influence of the polynomial degrees of the basis functions on the estimated error. The domain and material parameters are the same as in the previous example. Figure 6.8 shows the estimated error for different combinations of polynomial degree l for the displacement, and k for the plastic strain and the Lagrange multiplier for a uniform refinement of the mesh. We observe that the convergences rates are asymptotically the same. This is what can be anticipated from the a priori results in Chapter 3 due to the expected low regularity of the solution. In the same way as in the low order discretization, the adaptive refinement based on the error estimator $\eta_{P,\ell}$ yields higher convergence rates as can be seen in Figure 6.9. Moreover, we observe that these convergence rates increase proportional to the polynomial degrees.

As in the previous setting, we investigate the influence of the bulk parameter for different polynomial degrees. First, we focus on the spaces $W_h^{1,0}$. Figure 6.11 indicates that the rate of convergence is asymptotically the same for bulk parameters smaller than $\theta = 0.7$. For spaces $W_h^{3,2}$, we observe in Figure 6.10 that the convergence rate does increases for smaller bulk parameters until it rest asymptotically the same from approximately $\theta = 0.2$.



(a) (b)

Figure 6.4: Lagrange multipliers for the first and second yield surface: **(a)** $(\lambda_{0,P,\ell} : \lambda_{0,P,\ell})^{1/2}$, **(b)** $(\lambda_{1,P,\ell} : \lambda_{1,P,\ell})^{1/2}$.

The error estimate for the non conform discretization consists of four terms $\eta_{P,\ell}$, $\eta_{P,\mathrm{dev}} := \| \operatorname{dev}(\sigma(w_\ell) - \mathbb{H}p_\ell) - \sigma_y \lambda_{P,\ell}\|_0$, $\eta_{P,NC} := \|\lambda_{P,\ell} - \mu_P\|_0$ and $\eta_{P,\Psi} := |\Psi(p_\ell) - (\mu_P, \sigma_y p_\ell)_0|^{1/2}$. In Figure 6.12, we observe that asymptotic behavior of the contributions seem to be of the same order.

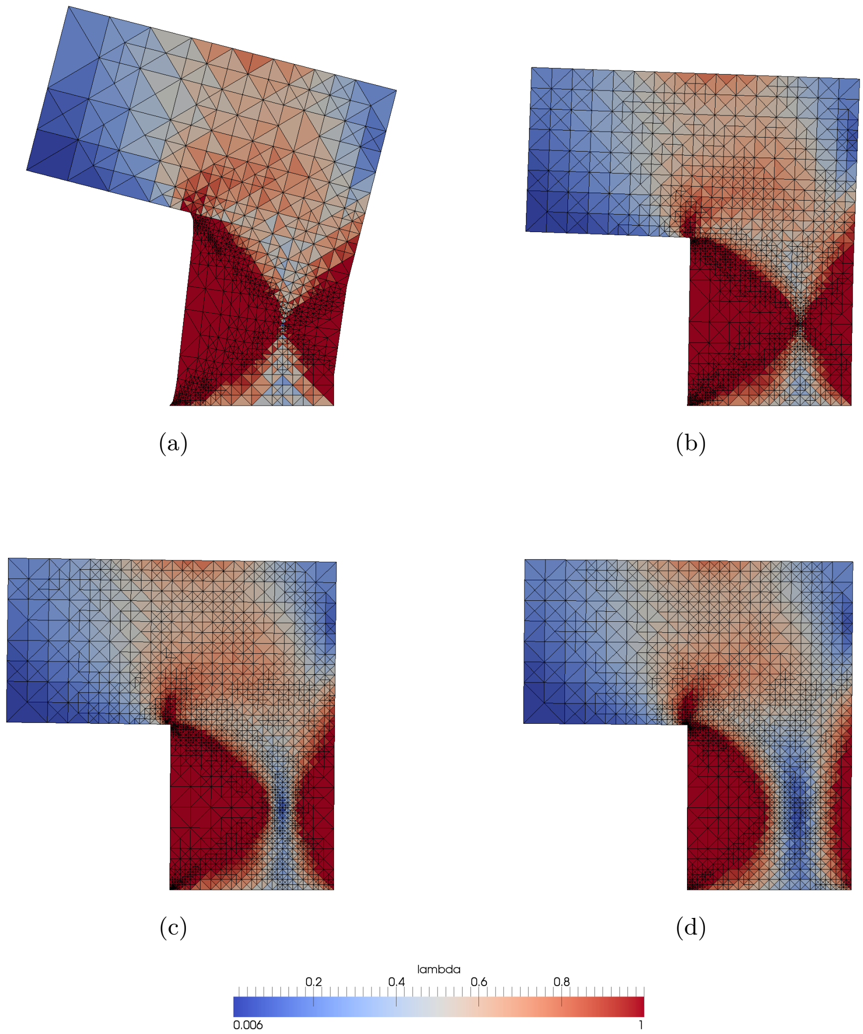


Figure 6.5: Displacement and norm of Lagrange multiplier for **(a)** $\xi = 0.1$, **(b)** $\xi = 1$, **(c)** $\xi = 10$, **(d)** $\xi = 100$.

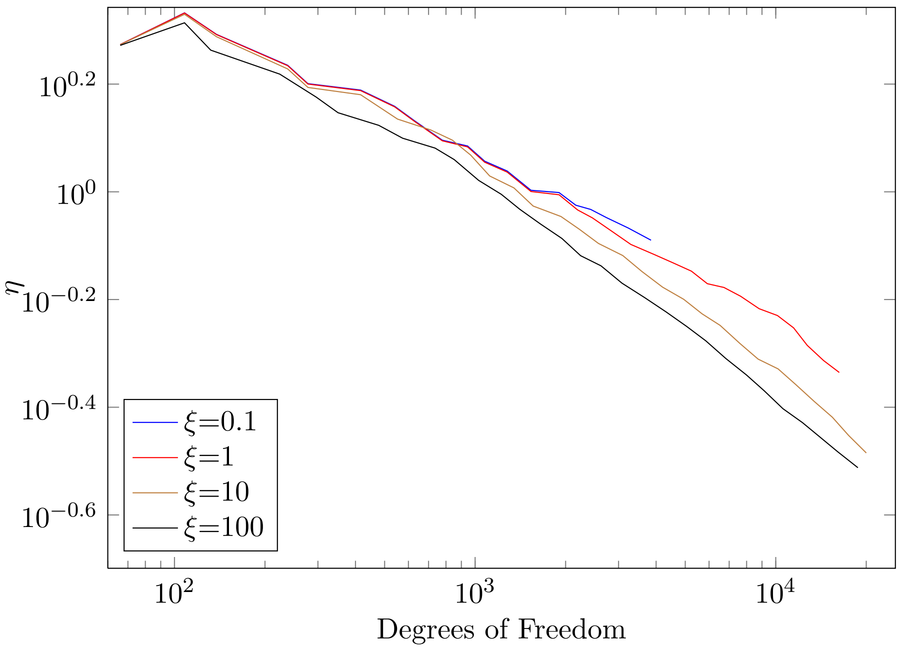


Figure 6.6: The estimated error for different values of the hardening parameter ξ.

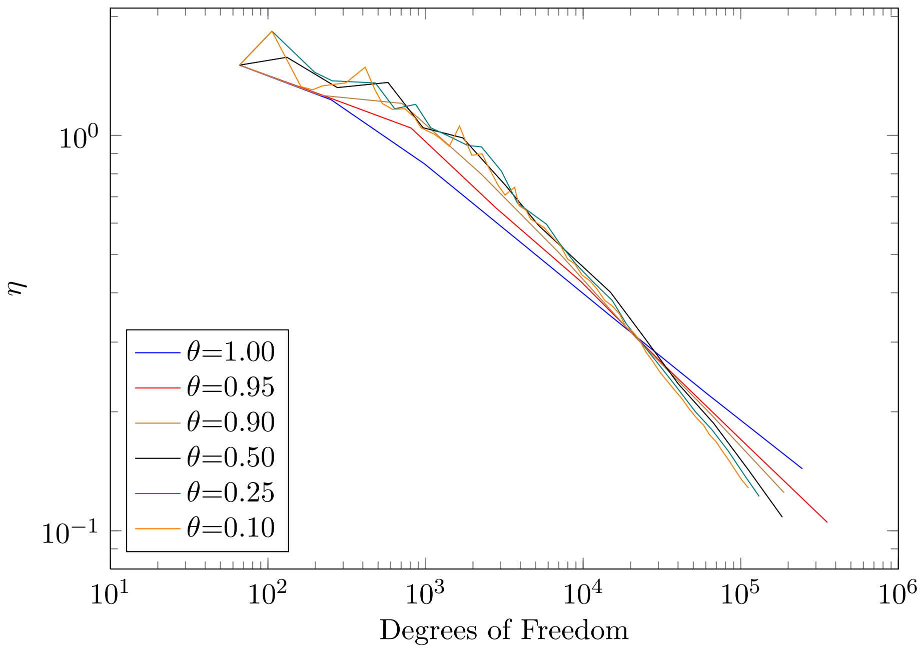


Figure 6.7: The estimated error for different values of the bulk parameter θ.

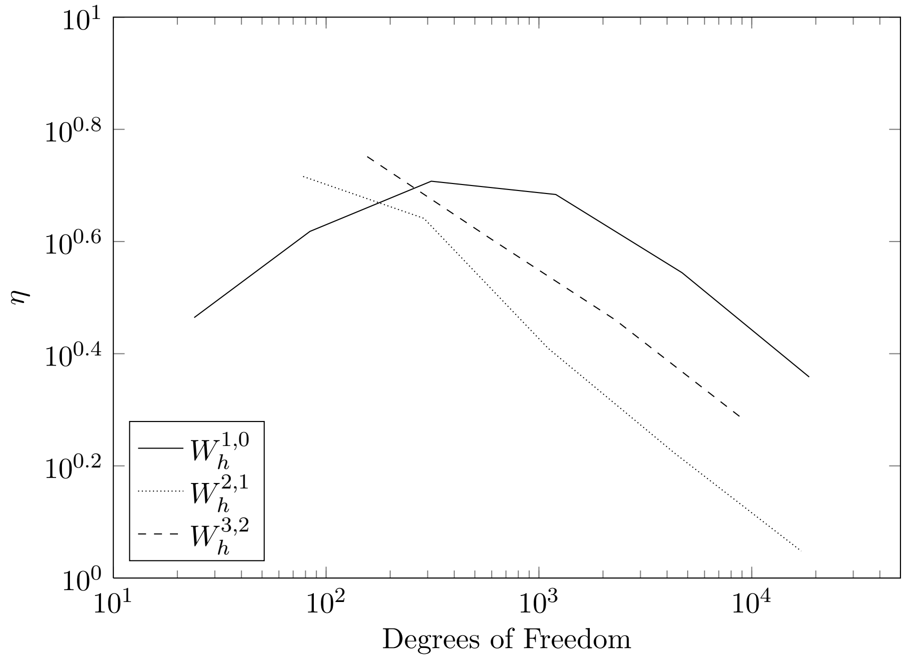


Figure 6.8: The estimated error for uniform mesh refinement and different polynomial degrees.

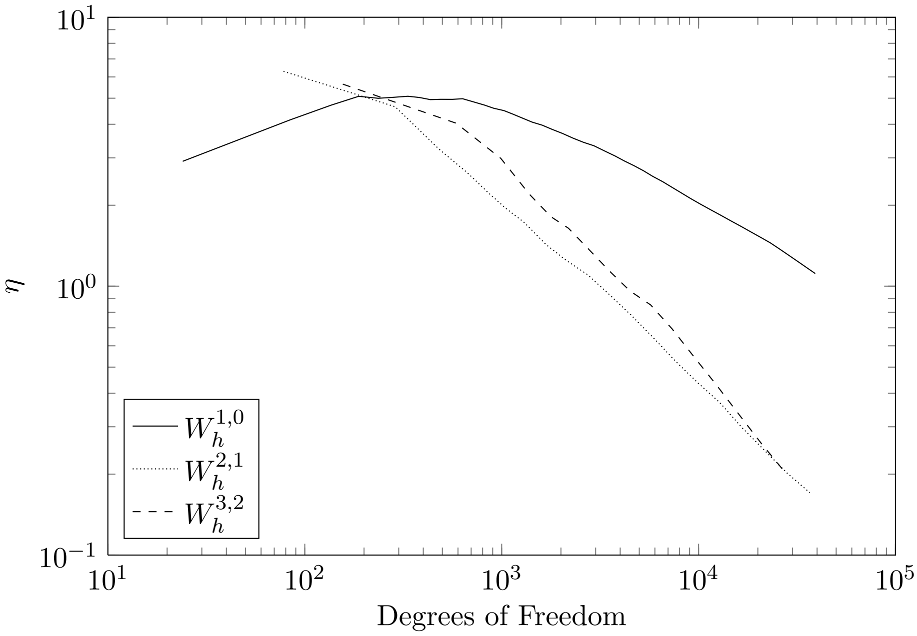


Figure 6.9: The estimated error for different polynomial degrees.

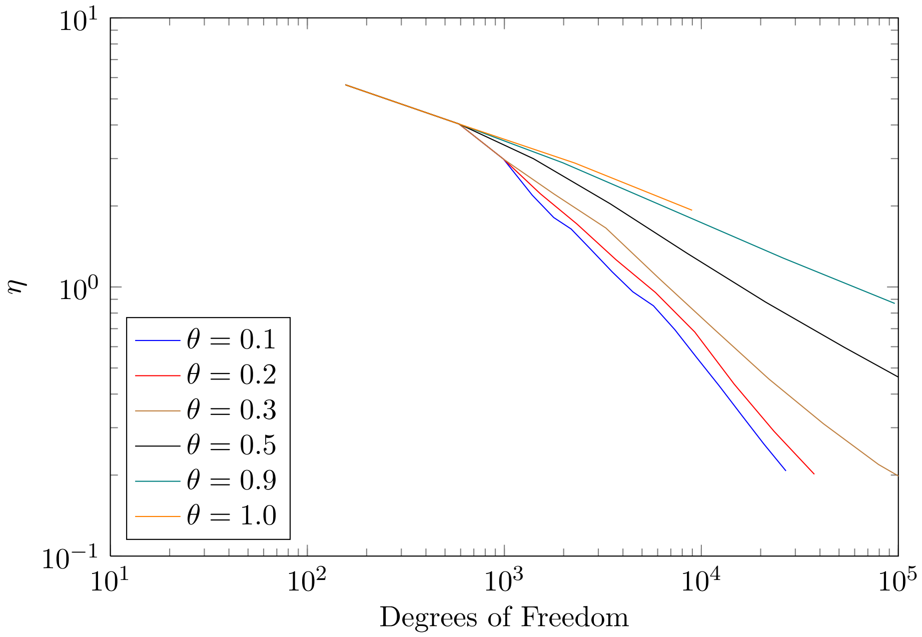


Figure 6.10: The estimated error for $W_h^{3,2}$ and different bulk parameters θ.

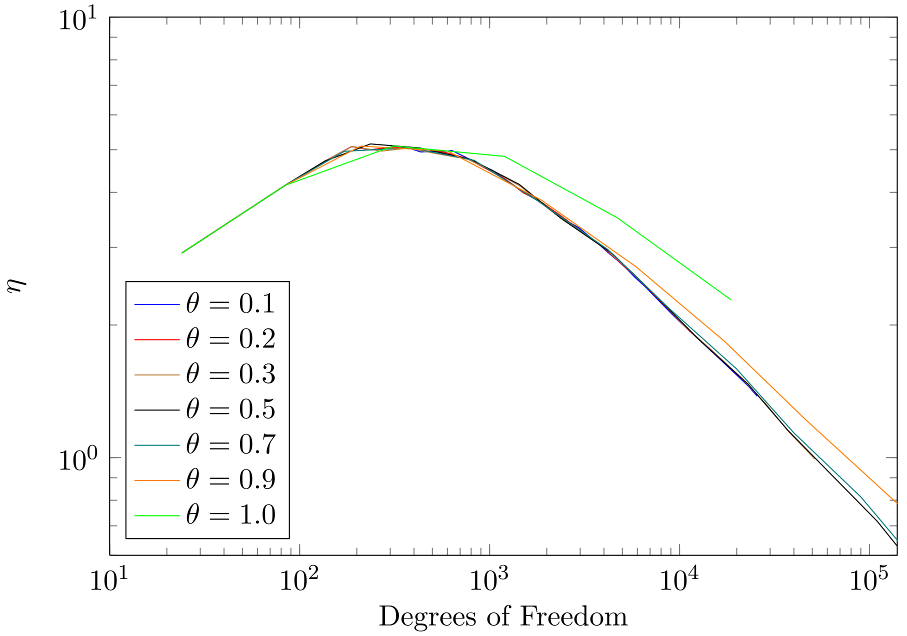


Figure 6.11: The estimated error for $W_h^{1,0}$ and different bulk parameters θ.

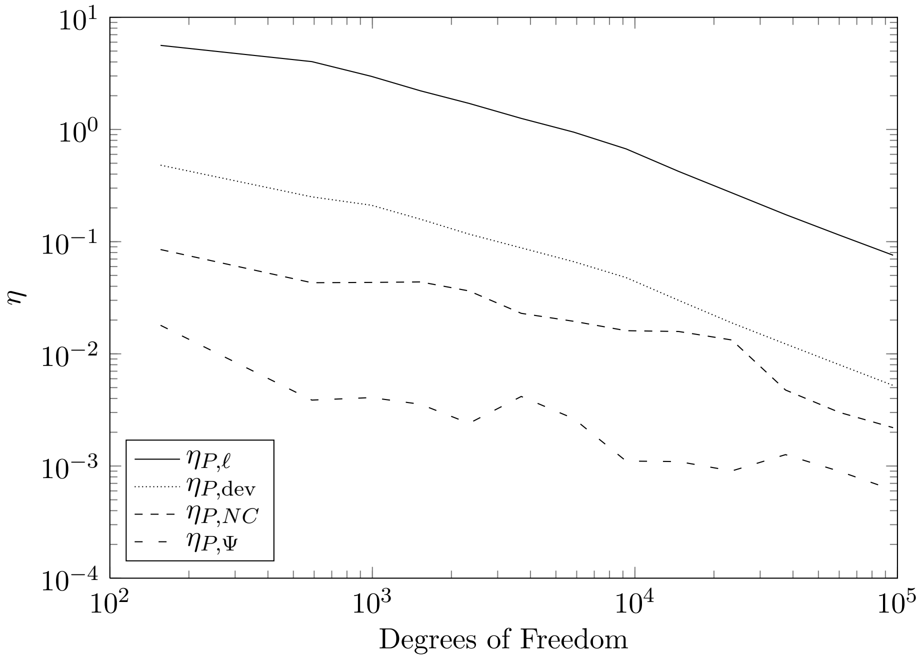


Figure 6.12: The contributions to the error estimator for $W_h^{3,2}$ and bulk parameter $\theta = 0.2$.

6.2 Frictional Contact

In this section, we present numerical results concerning the adaptive finite elements for the frictional contact problems, which have been introduced in Chapter 2. The numerical results were computed with the help of an Uzawa type algorithm similar to the one in the previous section. We note again that this is not the most efficient algorithm. However, it is easy to implement and that is the reason why we have chosen it here. The first example is given on the square $\Omega := [-1, 1] \times [-1, 1]$. As in the previous section, we assume the hardening tensor to be given by $\mathbb{H} = \xi\mathbb{I}$. The material parameters read $\lambda_{\mathbb{C}} = 1000$, $\mu_{\mathbb{C}} = 1000$, $\xi = 500$ and $\sigma_y = 10$. The domain is fixed at the bottom, i.e., we assume homogeneous Dirichlet data at $\Gamma_D := [-1, 1] \times \{0\}$. Moreover, we introduce a rigid foundation with a plane surface $\Psi(x) = 0.98$. The friction resistance reads $f_F = 15.5$. The surface traction $f_N(x, y) = 40y$ acts on the boundary part $\Gamma_N := \{-1\} \times [-1, 1]$. We choose bilinear function for the displacement and piecewise constant one for the plastic strain. In Figure 6.13, the estimated errors for adaptive and uniform refinement is plotted over the degrees of freedom. As expected, we observe that the adaptive scheme yields a better convergence rate compared to uniform mesh refinement.

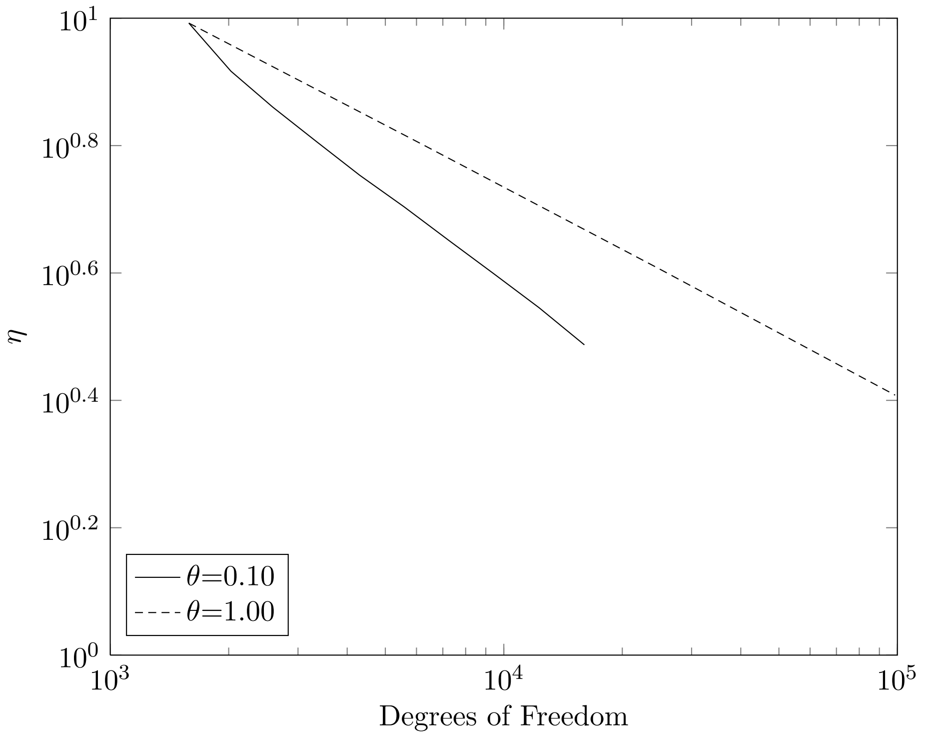


Figure 6.13: The estimated error for uniform $\theta = 1.0$ and adaptive $\theta = 0.1$ refinement.

In the second example, we return to the L-shape domain of this chapter's first section. The material parameters are the same as for the first example of this section. The surface of the rigid foundation is given by $\psi = (x - 0.5)^2 + 0.98$. We want to investigate the influence of friction. Therefore we compare frictionless contact and a friction resistance of $f_F = 15.5$. In Figure 6.14, we see the norm $(\lambda_P : \lambda_P)^{1/2}$ of the Lagrange multiplier for the dissipation functional and the displacement on a adaptively refined meshes after 9 steps of the adaptive scheme. The left subfigure shows the frictionless contact whereas the right one shows the influence of the same obstacle with friction. We observe that in the absence of friction the body slips slightly to the left beneath the obstacle. Moreover, the shape of the plastic zone changes significantly and therefore the adaptive refinements differ in some parts. For both settings the main refinements are observed towards the reentrant corner. In the frictionless problem the mesh has to resolve the vertical shear zone in the bottom which is similarly observed in the experiments of the previous section. If the friction resistance is positive, we observe that no shear zone has to be resolved but the contact boundary is further refined. Furthermore, the obstacle and the deformed domain after 13 refinement steps is depicted in Figure 6.15. The colors indicate the size of the norm of the plastic multiplier.

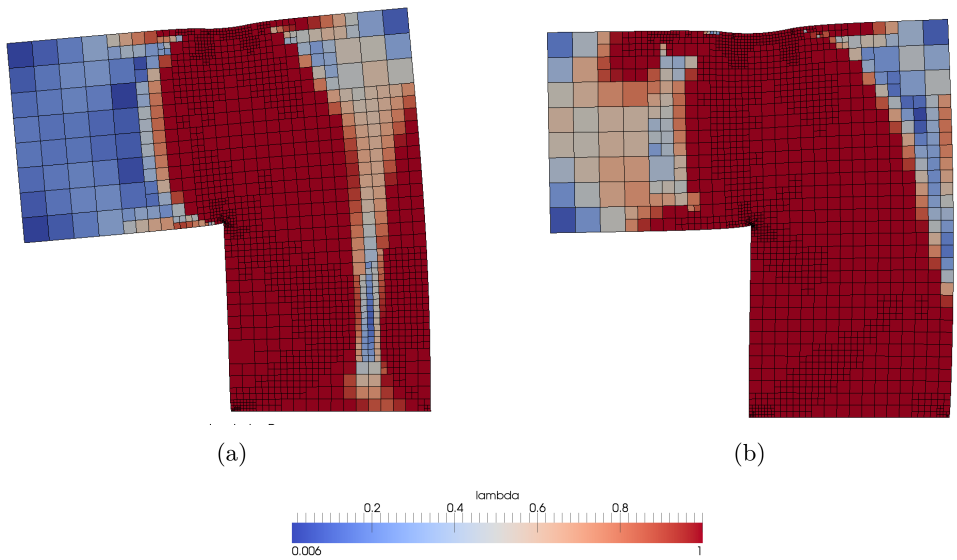


Figure 6.14: Displacement and norm of Lagrange multiplier for frictionless **(a)** and frictional **(b)** contact.

Finally, we turn again on the investigation of the influence of the polynomial degree on the rate of convergence. We consider the the frictional contact problem on the square $\Omega = [-1, 1]^2$ with homogeneous Dirichlet conditions on $\Gamma_D = [-1, 1] \times \{1\}$ and Neumann force $f_N = -10$ on $\{-1\} \times [-1, 1]$ and zero elsewhere. The obstacle

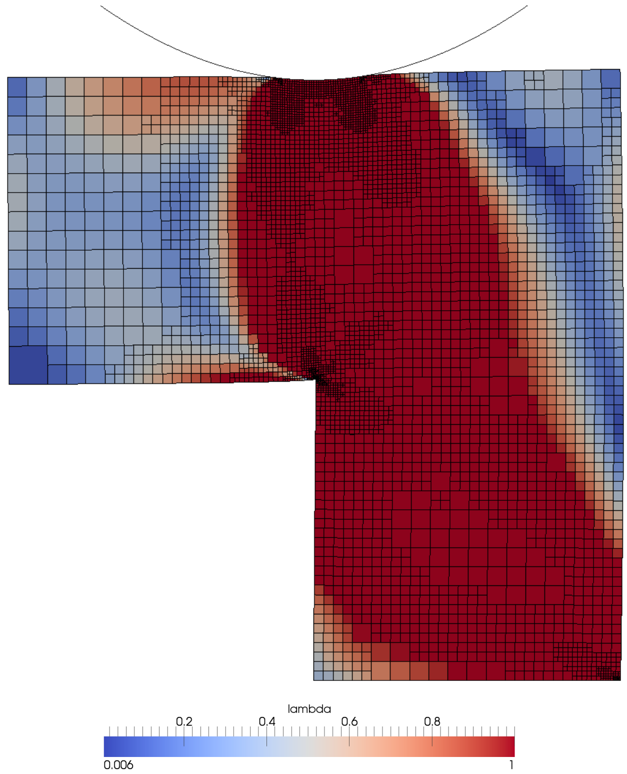


Figure 6.15: Obstacle and deformed domain for frictional contact. Colors indicate the norm of Lagrange multiplier

is prescribed by the function $\psi(x, y) = (1 - x^2) - 1.95)^{1/2}$. Moreover, the material parameters read $E = 2500$, $\nu = 0.25$, $\xi = 100$ and $\sigma_y = 45$ and the frictional resistance is given as $f_F(x, y) = 1.5 \exp(1 + x) - 1$. Figure 6.16 displays the mesh after after some adaptive refinements for different polynomial degrees. We observe that both meshes are refined towards the zones where the contact zone ends and to the upper corners where the boundary conditions change. However, the refinements for the discretization spaces $W_h^{2,1}$ are more focused on these expected points whereas for $W_h^{1,0}$ more inner cells are refined. This difference is somehow what we would expected since the solution should be more regular away from the singularities. Furthermore, we observe in Figure 6.17 that this results in a better convergence rate for $W_h^{2,1}$

The last Figure shows the contact of a rigid abrasive grain with a steel body. The solution was obtained by a regularization and the application of Newton's method. It was computed within the project “Mathematische Modellierung und effiziente Numerik zur Simulation vom Werkzeugschleifen” which was part of the priority program “Prognose und Beeinflussung der Wechselwirkungen von Strukturen und Prozessen” 1180 of the German research foundation (DFG).

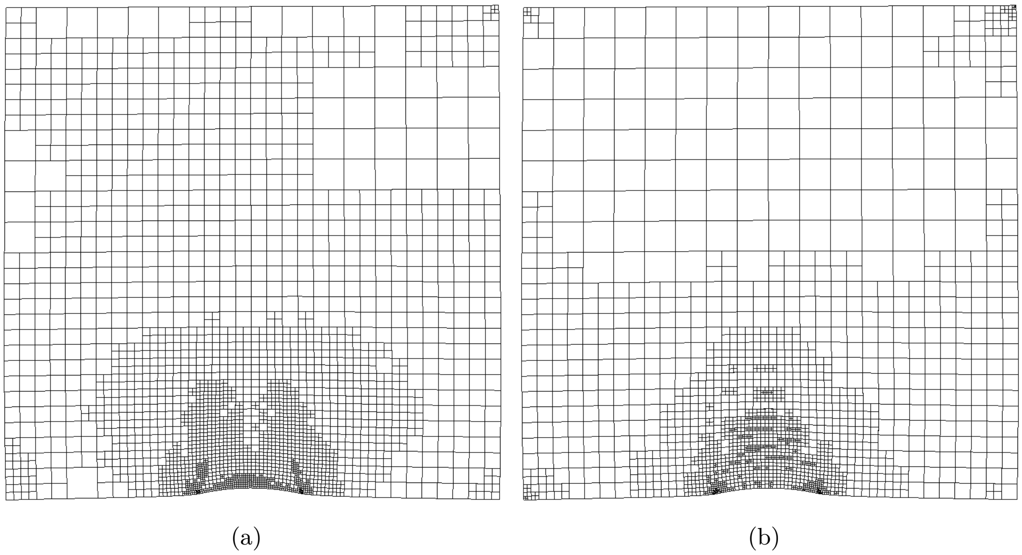


Figure 6.16: Displacement and grid for polynomial degrees **(a)** one, zero and **(b)** two, one.

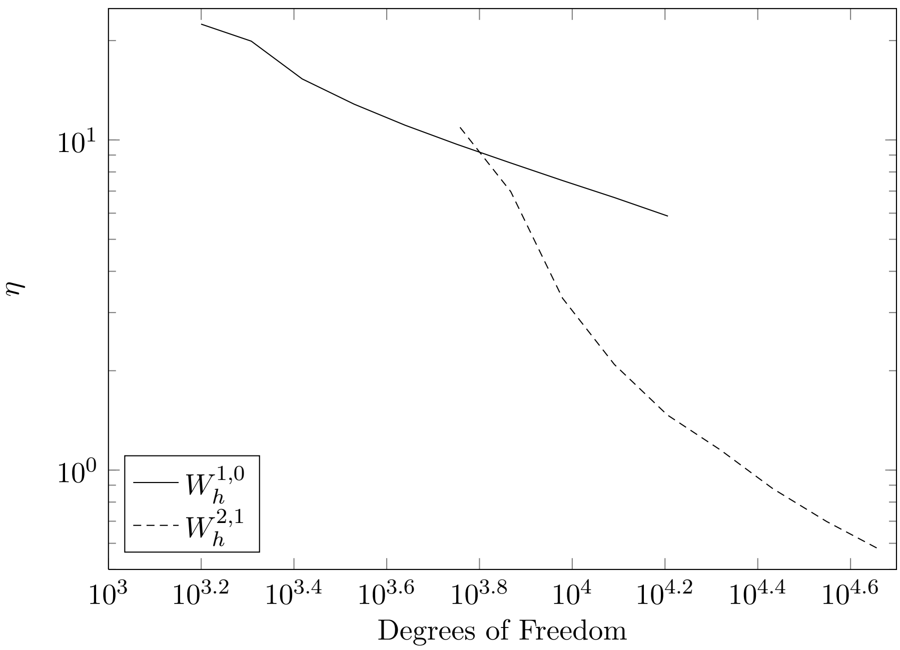


Figure 6.17: The estimated error for different polynomial degrees and adaptive refinements.

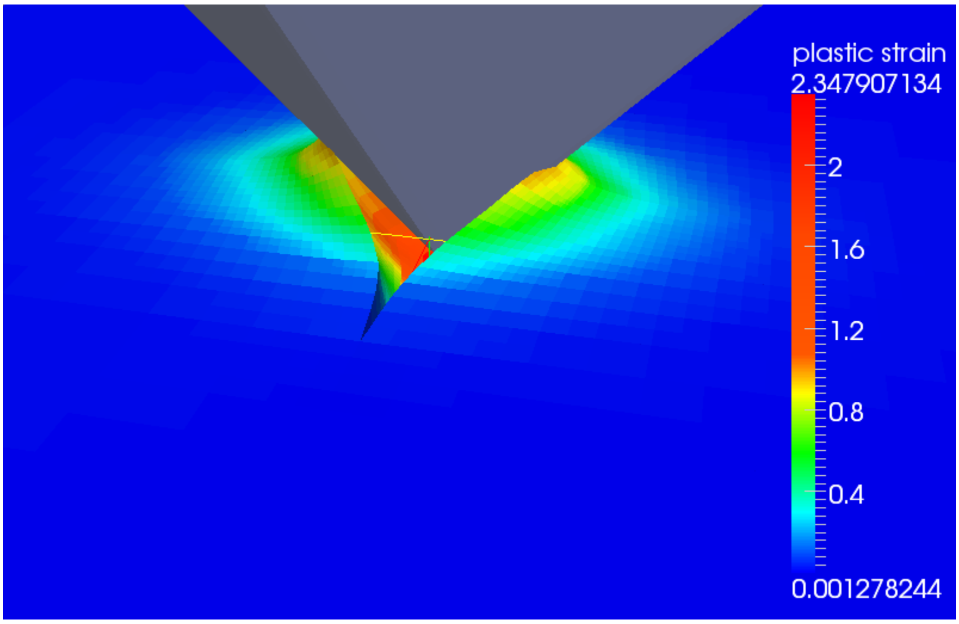


Figure 6.18: The norm of the plastic strain and the deformation of a steel body in contact with a rigid abrasive grain.

7 Outlook

In this chapter we give a brief outlook on some remaining open questions. We further try to give ideas, which could lead to some answers in the future.

One main result of this thesis is the derivation of error estimates and of convergence rates for the mixed schemes. Although the a priori results of Section 3.1 for arbitrary spaces hold in three spatial dimensions the convergence rates derived in Section 3.2 only hold in two dimensions. Moreover, they only hold for conform discretizations of the Lagrange multiplier for the dissipation functional. Hence, in practice the possible polynomial degree of the multiplier is restricted to be less or equal to one. The problem in the proof of the estimates for the Lagrange multiplier for friction in three dimensions and plasticity in two dimensions are somehow similar. It remains an open question if it is possible to overcome these problems for example with the help of the normality rule.

The proof of the convergence of the AFEM in Chapter 5 relies strongly on the fact that the material law is fulfilled pointwise and the set of Lagrange multipliers is conform. However numerical experiments suggest that also other conform adaptive schemes converge [97]. Additionally, the examples in Chapter 6 suggest that the conformity is not a necessary condition for convergence. It remains to investigate whether maybe techniques known from nonconforming [33] or mixed [37] methods for linear problems can be adapted to proof convergence.

Furthermore, the adaptive strategy of Chapter 5 is restricted to h-refinement. The use of hp-refinement is known to yield exponential convergence rates [9]. For the Poisson model problem, there exists even a convergence result [24]. The development of useful hp marking strategies for the elastoplastic contact problem is an interesting issue. Moreover, the proof of convergence for hp-FEM for nonlinear problems in general is still an open question. Whereas for discretizations of the Signorini problem with the boundary element method, a convergence result for two dimensional problems is found in [77].

The development of efficient numerical solution algorithms for problems in elastoplasticity with hardening is vast field. The finite element methods in this thesis define discrete optimization problems. It remains to employ more efficient algorithms than the Uzawa algorithm which computed the discrete solutions of Chapter 6. An approach of a cascadic multigrid method for contact problems was presented in [18] and adapted to accelerate projective SOR-procedures in [95]. It remains an open task to adapt the idea to the problems of this thesis and investigate its performance in this setting.

We have seen that it is also possible to use the mixed approach only for the contact conditions. Within an iterative approach the nonlinearity from the elastoplastic behavior can be treated via a direct Newton like scheme, see e.g. [44, 63]. This way at least the contact conditions could be resolved by highly efficient methods like the multigrid scheme proposed in [73, 72, 111]. The analysis of such iterative schemes based on different approaches for the different types of nonlinearities has yet to be accomplished. in addition, like for many modern mathematical methods, the problem of an implementation for real world problems remains only partly solved until now for the elastoplastic contact problems.

A different approach for the definition of sets of discrete Lagrange multipliers is the use of so-called biorthogonal basis functions. The choice ensures an orthogonality relation between the basis of the primal solution and the Lagrange multipliers. The choice of the multiplier for the dissipation functional already has the same properties. The result is an easy to handle discrete system. In [11] a similar approach has been used for the solution of the Signorini problem by a Newton like method. The approach is closely related to the mortar method found for example in [110].

The optimal convergence result of Section 5.3 only holds for conform lowest order finite elements on triangles. As discussed, the problems with the use of quadrilaterals are the same as in the linear case. However, it is no problem to conclude the optimal convergence if the results for the refinement of meshes based on quadrilaterals can be shown.

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